TEME DE ALGEBRA LINIARA

1. *DETERMINANTI VANDERMONDE*

$$V(a,b,c) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

$$V(a,b,c,d) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

$$V(a_{1,}, a_{2,\dots,} a_{n,}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} =$$

$$= \prod_{1 \le i < j \le n} (a_j - a_i)$$

Ex. 1.1 (164/2019)

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a+2 & b+2 & c+2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix}, \ a < b < c.$$

A: D= 0; ***B***: D ≤ 0 ; **C**: $D < 0$; ***D***: D > 0 ; **E**: D $=-a^2 - b^2 - c^2$

Rezolvare

Scazand linia 1 din linia 2, obtinem:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a+1 & b+1 & c+1 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix} = V(a+1, b+1, c+1) = (b+1-a-1)(c+1-a-1)(c+1-b-1) = (b-a)(c-a)(c-b) > 0,$$

$deci\ raspuns\ corect$ ***D*.**

Ex. 1.2 (188/2019, 188/2020)

Multimea solutiilor reale ale ecuatiei

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & -1 & 2 \\ 1 & x^3 & -1 & 8 \\ 1 & x^2 & 1 & 4 \end{vmatrix} = 0 \text{ este:}$$

A: {-1,1,2} ; ***B***: $\mathbb{R}\backslash\{-1,1,2\}$; **C**: $\{-1,1-2\}$; ***D***: Φ ; **E**: D {1}

Rezolvare

Schimband intre ele liniile 3 si 4, obtinem:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & -1 & 2 \\ 1 & x^3 & -1 & 8 \\ 1 & x^2 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & -1 & 2 \\ 1 & x^2 & 1 & 4 \\ 1 & x^3 & -1 & 8 \end{vmatrix} = -V(1, x, -1, 2) =$$

$-1(x-1)(-1-1)(2-1)(-1-x)(2-x)(2+1) = 6(x-1)(x+1)(2-x) = 0,\ deci\ x \in \{-1,1,2\}$. Raspuns corect: **A.**

Ex. 1.3 (171/2019)

Determinantul $\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{vmatrix}$ este:

A: 16i; ***B***: $-$ 16i ; **C**: 16; ***D***: -16 ; **E**: 0

Rezolvare

$\Delta = V(1, -i, -1, i) = (-i-1)(-1-1)(i-1)(-1+i)(i+i)(i+1) = 16i$; raspuns corect **A.**

2. *ECUATIA CAYLEY-HAMILTON (C-H)*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{C})$$

$$\xRightarrow{calcul} \begin{cases} A^2 - (a+d)A + (ad-bc)I_2 = 0_2 \\ A^2 - (TrA)A + \det(A)I_2 = 0_2 \end{cases}$$

Ex. 2.1 (Admitere 2009)

Fie $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{Z})$.

(i) Numarul solutiior ecuatiei $X^2 = A$, $X \epsilon \mathcal{M}_2(\mathbb{R})$este:

A: 1; ***B***: 0 ; **C**: 2; ***D***: 3; **E**: ∞

(ii) Numarul solutiior ecuatiei $X^3 = A$, $X \epsilon \mathcal{M}_2(\mathbb{Z})$este:

A: 1; ***B***: 5 ; **C**: 2; ***D***: 0; **E**: ∞

(iii) Daca $A^3 = a_3 A + b_3 I_2$, atunci perechea (a_3, b_3) este:

A: $(1,5)$; ***B***: $(21,25)$; **C**: $(27,10)$; ***D***: $(27,15)$; **E**: $(25,10)$

(iv) Daca $A^n = a_n A + b_n I_2$, atunci :

A: $a_{n+1} = 5a_n + b_n$; ***B***: $a_{n+1} = 5a_n + 2b_n$; **C**: $a_{n+1} = 3a_n + b_n$; ***D***: $a_{n+1} = -5a_n + 2b_n$; **E**: $b_{n+1} = 3a_n$

Rezolvare

(i) $X^2 = A \xRightarrow{propr} \det(X^2) = \det(A) \xRightarrow{propr} (\det X)^2 = -2$, imposibil, deoarece $X \epsilon \mathcal{M}_2(\mathbb{R})$, deci $detX \epsilon \mathbb{R}$ si $(\det X)^2 \geq 0$.

(ii) $X^3 = A \xRightarrow{propr} \det(X^3) = \det(A) \xRightarrow{propr} (\det X)^3 = -2$, imposibil. Intr-adevar, $X \epsilon \mathcal{M}_2(\mathbb{Z})$, deci $detX \epsilon \mathbb{Z}$; pe de alta parte, din $(\det X)^3 = -2$, rezulta $detX = \sqrt[3]{-2} \notin \mathbb{Z}$.

(iii) (C-H): $A^2 - (1+4)A + (4-6)I_2 = 0_2$, adica $A^2 - 5A - 2I_2 = 0_2 \Leftrightarrow A^2 = 5A + 2I_2$ **(1)**.

Mai departe, utilizand relatia **(1)**, obtinem: $A^3 = A^2 A = 5A^2 + 2A = 5(5A + 2I_2) + 2A = 27A + 10\, I_2$, deci $(a_3, b_3) = (27,10)$.

(iv) Din datele problemei, primim:

$A^n = a_n A + b_n I_2$ **(2)**

$A^{n+1} = a_{n+1} A + b_{n+1} I_2$ **(3)**

Mai departe, utilizand relatiile **(2)** si **(1)**, obtinem:

$A^{n+1} = A^n A = a_n A^2 + b_n A = a_n(5A + 2I_2) + b_n A$, deci

$A^{n+1} = (5a_n + b_n)A + 2a_n I_2$ **(4)**.

Din **(3)** si **(4)** rezulta $a_{n+1} = (5a_n + b_n)$ si $b_{n+1} = 2b_n$.

3. *PUTEREA* n *A UNEI MATRICE PATRATICE*

3.1. Matrice de rotatie

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R}) \overset{ind}{\Longrightarrow}$$

$$A^n = \begin{pmatrix} \cos nt & -\sin nt \\ -\sin nt & \cos nt \end{pmatrix}, n \epsilon \mathbb{Z}$$

Generalizare

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R}) \ ; \ r = \sqrt{a^2 + b^2}$$

$$A = r \begin{pmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{pmatrix}$$

Deoarece $(\frac{a}{r})^2 + (\frac{b}{r})^2 = 1$, exista $t \epsilon [0,2\pi)$ astfel incat

$\frac{a}{r} = \cos t \ si \ \frac{b}{r} = \sin t,$ deci

$A = r\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, asadar $A^n = r^n \begin{pmatrix} \cos nt & -\sin nt \\ \sin nt & \cos nt \end{pmatrix}$.

Ex. 3.1 (Admitere 2010)

Fie $A = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R})$ si $A^n = \begin{pmatrix} a_n & -b_n \\ b_n & a_n \end{pmatrix}$, $n \epsilon \mathbb{N}$.

(i) A^2 este:

A: 2A; ***B***: $2A + 4I_2$; **C**: I_2; ***D***: 0_2; E: $2A - 4I_2$

(ii) A^{48} este:

A: $2^{24}I_2$; ***B***: $-2^{24}I_2$; **C**: I_2; ***D***: $2^{48}I_2$; **E**: $-2^{48}I_2$

(iii) $\dfrac{a_{20}^2+b_{20}^2}{a_{10}^2+b_{10}^2}$ este:

A: 2^{15}; ***B***: 2^5; **C**: 1; ***D***: 2^{10}; E: 2^{20}

(iv) Numarul valorilor n ≥ 1 pentru care $A^n = 8I_2$ este:

A: 3; ***B***: 0; **C**: 1; ***D***:2; **E**: ∞

Rezolvare

(i) Din ecuatia (C-H) obtinem:

$$A^2 - 2A + 4I_2 = 0_2 \Leftrightarrow A^2 = 2A - 4I_2 \ .$$

(ii) ***Metoda 1.*** Scriem A sub forma unei matrice de rotatie generalizate:

$$A = \sqrt{1+3}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = 2\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix},$$ de unde obtinem:

$$A^n = 2^n \begin{pmatrix} \cos\dfrac{n\pi}{3} & -\sin\dfrac{n\pi}{3} \\ \sin\dfrac{n\pi}{3} & \cos\dfrac{n\pi}{3} \end{pmatrix}$$

Astfel, $A^{48} = 2^{48}\begin{pmatrix} \cos 16\pi & -\sin 16\pi \\ \sin 16\pi & \cos 16\pi \end{pmatrix} = 2^{48}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$ $2^{48}\, I_2$.

Metoda 2. Din ecuatia (C-H) rezulta

$A^2 - 2A + 2^2 I_2 = 0_2$, de unde obtinem:

$A^3 + (2I_2)^3 = (A + 2I_2)(A^2 - 2A + 2^2 I_2) = (A + 2I_2)0_2 = 0_2$, deci $A^3 = (-2I_2)^3 = -8I_2$.

Astfel, $A^{48} = (A^3)^{16} = (-2^3 I_2)^{16} = 2^{48}\, I_2$.

(iii) ***Metoda 1.*** Din datele problemei si **(ii)**, **Metoda 1**, primim:

$a_n = 2^n \cos\frac{n\pi}{3}$ si $b_n = 2^n \sin\frac{n\pi}{3}$, asadar

$a_n^2 + b_n^2 = 2^{2n}$. Astfel, $\frac{a_{20}^2+b_{20}^2}{a_{18}^2+b_{18}^2} = \frac{2^{40}}{2^{20}} = 2^{20}$.

Metoda 2. Observam ca $\det(A^n) = a_n^2 + b_n^2$, deci

$$\frac{a_{20}^2+b_{20}^2}{a_{18}^2+b_{18}^2} = \frac{\det(A^{20})}{\det(A^{10})} = \frac{(\det A)^{20}}{(\det A)^{10}} = (\det A)^{10} = 2^{20}.$$

(iv) ***Metoda 1.***

$$A^n = 8I_2 \overset{(ii),\ Met.1}{\Longleftrightarrow} 2^n \begin{pmatrix} \cos\frac{n\pi}{3} & -\sin\frac{n\pi}{3} \\ \sin\frac{n\pi}{3} & \cos\frac{n\pi}{3} \end{pmatrix} = 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Leftrightarrow 2^n \cos\frac{n\pi}{3} = 8 \wedge 2^n \sin\frac{n\pi}{3} = 0 \Leftrightarrow 2^n \cos\frac{n\pi}{3} = 8 \wedge \frac{n\pi}{3} = k\pi,\ k \in \mathbb{N} \Leftrightarrow n = 3k,\ 2^{3k}\cos(k\pi) = 8 \Leftrightarrow 2^{3k}(-1)^k = 8 \Leftrightarrow (-8)^k = 8,\ k \in \mathbb{N}$, ceea ce este imposibil, deoarece $8 \notin \{(-8)^k \,/\, k \in \mathbb{N}\}$.

Metoda 2. Din (ii), Metoda 2, rezulta $A^3 = -8I_2$, de unde deducem

$$\begin{cases} A^{3k} = (-8)^k I_2 \\ A^{3k+1} = (-8)^k A \\ A^{3k+2} = (-8)^k A^2 \end{cases}$$

In nici una dintre aceste situatii nu putem primi $A^n = 8I_2$.

3.2. Utilizarea ecuatiei Cayley-Hamilton (C-H)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{C})$$

$$\overset{calcul}{\Longrightarrow} \begin{cases} A^2 - (a+d)A + (ad-bc)I_2 = 0_2 \\ A^2 - (TrA)A + \det(A)I_2 = 0_2 \end{cases}$$

3.2.1. Daca ***det A = 0***, atunci $A^2 = (TrA)A = (a+d)A$ $\overset{ind}{\Longrightarrow} A^n = (a+d)^{n-1}A = (TrA)^{n-1}A,\ n \geq 2.$

3.2.2. Daca ***TrA = a+d = 0***, atunci $A^2 = (bc-ad)I_2 \overset{ind}{\Longrightarrow}$

$$A^{2n} = (bc-ad)^n I_2 \qquad si$$

$$A^{2n+1} = (bc-ad)^n A$$

3.3. Utilizarea unor formule de calcul matriceal

Fie A, B $\boldsymbol{\epsilon \mathcal{M}_k}(\mathbb{C})$***. Daca AB=BA, atunci:***

$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + B^{n-1})$$

$$A^{2n+1} - B^{2n+1} = (A+B)(A^{2n} - A^{2n-1}B + A^{2n-2}B^2 - A^{2n-3}B^3 + \cdots + B^{2n})$$

$(A+B)^n = \sum_{k=0}^{n} C_n^k\, A^{n-k}B^k$,

unde $n \geq 1$ si $A^0 \overset{\text{def}}{=} I_k$.

OBS. Egalitatile de mai sus au loc daca, de exemplu, $B = \alpha I_k,\ \alpha \in \mathbb{C}$

3.3.1. Daca $A^2 + \alpha A + \alpha^2 I_k = 0_k$, atunci $A^3 = \alpha^3 I_k$, deci

$$\begin{cases} A^{3n} = \alpha^{3n} I_k \\ A^{3n+1} = \alpha^{3n} A \\ A^{3n+2} = \alpha^{3n} A^2 \end{cases}$$

Intr-adevar,

$$A^3 - (\alpha I_k)^3 = (A - \alpha I_k)(A^2 + \alpha A + \alpha^2 I_k) = 0_k$$

3.3.2. Daca $A^2 - \alpha A + \alpha^2 I_k = 0_k$, atunci $A^3 = -\alpha^3 I_k$, deci

$$\begin{cases} A^{3n} = (-1)^n \alpha^{3n} I_k \\ A^{3n+1} = (-1)^n \alpha^{3n} A \\ A^{3n+2} = (-1)^n \alpha^{3n} A^2 \end{cases}$$

Similar,

$$A^3 + (\alpha I_k)^3 = (A + \alpha I_k)(A^2 - \alpha A + \alpha^2 I_k) = 0_k$$

3.3.3. Daca $A^n + \alpha A^{n-1} + \alpha^2 A^{n-2} + \cdots + \alpha^n I_k = 0_k$, atunci $A^{n+1} = \alpha^{n+1} I_k$

3.3.4 Daca $A^{2n} - \alpha A^{2n-1} + \alpha^2 A^{2n-2} - \alpha^3 A^{2n-3} + \cdots + \alpha^{2n} I_k = 0_k$, atunci $A^{2n+1} = -\alpha^{2n+1} I_k$

Ex. 3.2 (Admitere 2006; 180,181,182/20200)

Fie $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \epsilon\ \mathcal{M}_2(\mathbb{R})$.

(i) Numarul solutiilor inversabile $X \epsilon\ \mathcal{M}_2(\mathbb{R})$ ale ecuatiei $X^{2006} = A$ este:

A: 2006; ***B***: 1003; **C**: 0; ***D***: ∞; **E**: 2

(ii) Numarul solutiilor $X \epsilon\ \mathcal{M}_2(\mathbb{R})$ ale ecuatiei $X^{2006} = A$ este:

A: 2006; ***B***: 1003; **C**: 0; ***D***: ∞; **E**: 2

(iii) Numarul solutiilor $X \epsilon\ \mathcal{M}_2(\mathbb{C})$ ale ecuatiei $X^{2006} = A$ este:

A: 2006; ***B***: 1003; **C**: 0; ***D***: ∞; **E**: 2

Rezolvare

(i) Trecand la determinant, similar cu **Ex.2.1**, primim ***det(X)=0,*** deci nu exista inversa matricei X.

(ii) Deoarece *det(X)=0*, in conformitate cu ***3.2.1.,*** obtinem

$$X^n = (a+d)^{n-1}X = (TrA)^{n-1}X\ ,\ n \geq 2,$$

unde $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Astfel, ecuatia $X^n = A$ devine

$$(a+d)^{n-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\ ,$$

adica $\begin{cases} a(a+d)^{n-1} = 2 \\ b(a+d)^{n-1} = 3 \\ c(a+d)^{n-1} = 4 \\ d(a+d)^{n-1} = 6 \end{cases}$

Adunand prima si ultima ecuatie, obtinem

$$(a+d)^n = 8 \Leftrightarrow (TrX)^n = TrA\,.$$

Astfel, pentru **(ii),** rezulta $(a+d)^{2006} = 8,$ cu $a, d \in \mathbb{R}$**,** asadar $a + d = \pm\sqrt[2006]{8}$, ceea ce conduce la doua solutii (a,b,c,d) ale sistemului de mai sus, deci la doua matrice X.

(iii) Similar cu **(ii)**, ecuatia binoma $(a+d)^{2006} = 8,$ cu $a, d \in \mathbb{C}$**,** are ***2006*** solutii complexe, ceea ce conduce la ***2006*** solutii (a,b,c,d) ale sistemului de mai sus, deci la ***2006*** matrice X.

Ex. 3.3 (Admitere 2016; 229/2020)

Fie $A \in \mathcal{M}_2(\mathbb{R})$ o matrice nenula astfel incat $A^{2016} = 0_2$. Cardinalul multimii $\{A^n / n \geq 1\}$ este:

A: 2016; ***B***: 1008; **C**: 0; ***D***:2; **E**: ∞

Rezolvare

Din $A^{2016} = 0_2$ rezulta $\det(A^{2016}) = \det(0_2),$

deci $(detA)^{2016} = 0$ i.e. $detA = 0.$ In conformitate cu ***3.2.1.*** si ipoteza $A^{2016} = 0_2$, obtinem $A^{2016} = (TrA)^{2015} A = 0_2.$ Deoarece $A \neq 0_2,$ deducem TrA=0.

Din relatiile det A =Tr A = 0, utilizand ecuatia (C-H)

$$A^2 - (TrA)A + \det(A) I_2 = 0_2\ ,$$

deducem $A^2 = 0_2$, deci $A^n = 0_2, \forall n \geq 2.$

Astfel, $M = \{A, A^2\} = \{0_2, A\},$ asadar M are 2 elemente.

Ex. 3.4 (Test admitere ; 669/2020)

Numarul solutiilor $X\epsilon\, \mathcal{M}_2(\mathbb{R})$ ale ecuatiei $X^{25} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ este:

A: 2; ***B***: 0; **C**: 10; ***D***:25; **E**: ∞

Rezolvare

Similar cu ***Ex. 3.2*** primim $\det(X) = det\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$=0 si $(\text{Tr}X)^{25} = Tr\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = 0$, i.e. $TrX = 0$. Din ecuatia (C-H) rezulta X^2=0_2, deci X^{25}=0_2, ceea ce contrazice ipoteza problemei $X^{25} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$.

Asadar, ***nu exista*** nici o matrice $X\epsilon\, \mathcal{M}_2(\mathbb{R})$ care sa fie solutie a ecuatiei date.

Ex. 3.5. (Simulare 2018; 768/2020)

Fie $A\epsilon\, \mathcal{M}_2(\mathbb{R})$ si $\lambda \,\epsilon\, \mathbb{R}$ astfel incat $A^2 - \lambda A + \lambda^2 I_2 = 0_2$.

Matricea A^{2018} este:

A: $\lambda^{2018} I_2$; ***B***: A; **C**: $\lambda^{2016} A^2$; ***D***: $\lambda^2 A^2$; **E**: 0_2

Rezolvare

Avem $A^3 + (\lambda I_2)^3 = (A + \lambda I_2)\,(A^2 - \lambda A + \lambda^2 I_2) =$

$(A + \lambda I_2)\, 0_2 = 0_2$, deci $A^3 = -(\lambda I_2)^3 = -\lambda^3 I_2$.

Mai departe , $A^{2018} = A^{2016} A^2 = (\,A^3)^{672}\, A^2 = (-\lambda^3 I_2)^{672}\, A^2 = (-1)^{672}\, (\lambda^3)^{672}\, (I_2)^{672}\, A^2 = \lambda^{2016}\, I_2 A^2 = \lambda^{2016}\, A^2$.

Ex. 3.6. (Admitere 2011; 172/2020))

Fie $A = \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$.

(i) Numarul $a \in \mathbb{R}$ pentru care $aA - A^2 = I_2$ este:

A: 1; ***B***: 3; **C**: 0; ***D***: 2; **E**: -1

(ii) $A + A^{-1}$ este:

A: $2I_2$; ***B***: **3**A ; **C**: A^2; ***D***: 0_2; **E**: Nu exista A^{-1}

(iii) $A^{2011} + (A^{-1})^{2011}$ este:

A: $A + I_2$; ***B***: **3**A ; **C**: A^2; ***D***: 0_2; **E**: : $2I_2$

Rezolvare.

(i) Ecuatia (C-H) este: $A^2 - 2A + I_2 = 0_2$, i.e.

$2A - A^2 = I_2$, deci $a = 2.$

(ii) Din $A^2 - 2A + I_2 = 0_2$, prin inmultire cu A^{-1} (inversa exista deoarece $\det A = 1 \neq 0$), rezulta $A^2A^{-1} - 2AA^{-1} + I_2A^{-1} = 0_2$, deci $A - 2I_2 + A^{-1} = 0_2$.

Astfel, $A + A^{-1} = 2I_2$.

(iii) Din $A + A^{-1} = 2I_2$, prin ridicare la patrat, obtinem

$A^2 + (A^{-1})^2 + 2AA^{-1} = 4I_2$, deci

$A^2 + (A^{-1})^2 = 2I_2$.

Se demonstreaza prin metoda inductiei matematice relatia

$A^n + (A^{-1})^n = 2I_2, \forall n \geq 1.$

Ex. 3.7. (Admitere 2019; 845/2020)

Fie A $\epsilon \mathcal{M}_2(\mathbb{C})$ o matrice inversabila astfel incat $A + A^{-1} = I_2$.

Matricea $I_2 + A + A^2 + A^3 + \ldots + A^{2019}$ este:

A: **2**A $- I_2$; ***B***: **2**A $+ I_2$; **C**: $-$ **2**A $+ I_2$; ***D***: $-$**2**A $- I_2$; **E**: A $+ I_2$

Rezolvare.

Inmultind egalitatea data cu A, obtinem: $A^2 + I_2 = A,$ deci

$A^2 = A - I_2$ **(1)**

De aici, obtinem :

$$A^3 = A^2.A \overset{(1)}{\Longrightarrow} A^3 = (A - I_2)A = A^2 - A \overset{(1)}{\Longrightarrow} A^3 = -I_2$$

Obs. Egalitatea $A^3 = -I_2$ se putea deduce, similar cu ***Ex. 3.5***, astfel: $A^3 + I_2 = A^3 + (I_2)^3 = (A + I_2)(A^2 - A + I_2) = (A + I_2).0_2 = 0_2,$ utilizand **(1)**.

Asadar,

$$A^3 = -I_2 \Rightarrow A^4 = A^3.A = -A,\ A^5 = -A^2,\ A^6 = I_2 \quad \textbf{(2)}$$

Astfel, din (2) rezulta:

$$I_2 + A + A^2 + A^3 + A^4 + A^5 = I_2 + A + A^2 - I_2 - A - A^2 = 0_2 \quad \textbf{(3)}$$

Mai departe obtinem, succesiv:

$I_2 + A + A^2 + A^3 + \ldots + A^{2019} = (I_2 + A + A^2 + A^3 + A^4 + A^5) + (A^6 + A^7 + A^8 + A^9 + A^{10} + A^{11}) + \cdots + (A^{2010} + A^{2011} + A^{2012} + A^{2013} + A^{2014} + A^{2015}) + (A^{2016} + A^{2017} + A^{2018} + A^{2019}) = (I_2 + A + A^2 + A^3 + A^4 + A^5) + A^6(I_2 + A + A^2 + A^3 + A^4 + A^5) + \cdots + A^{2010}(I_2 + A + A^2 + A^3 + A^4 + A^5) + A^{2016}(I_2 + A + A^2 + A^3)$

$\xrightarrow{(3)\&(2)} I_2 + A + A^2 + A^3 + \ldots + A^{2019} = 0_2 + A^6.0_2 + \cdots + A^{2010}.0_2 + (A^6)^{372}(I_2 + A + A^2 + A^3) = I_2(I_2 + A + A^2 + A^3) = I_2 + A + A^2 + A^3$

$\xrightarrow{(1)\&(2)} I_2 + A + A^2 + A^3 + \ldots + A^{2019} = I_2 + A + A^2 + A^3 = I_2 + A + A - I_2 - I_2 = 2A - I_2.$

Ex. 3.8. (Simulare 2019; 819,829,821/2020)

Fie ε radacina pozitiva a ecuatiei $x^2 - x - 1 = 0$ si

$$A = \begin{pmatrix} 1-\varepsilon & \frac{1}{\varepsilon} \\ 1 & \varepsilon - 1 \end{pmatrix} \epsilon\, \mathcal{M}_2(\mathbb{R}).$$

(i) ε^3 este:

A: $\varepsilon - 2$; ***B***: **2**$\varepsilon - 1$; **C:** **2**$\varepsilon + 1$; ***D***: $-\varepsilon + 2$: 2; **E**: ε

(ii) det(A^{2019}) este:

A: 1; ***B***: 0; **C**: 2019; ***D***: -1; **E**: 2

(iii) Matricea A^{2019} este:

A: εI_2; ***B***: $-$ A; **C**: I_2; ***D***: $-\varepsilon I_2$; **E**: A

Rezolvare.

(i) Din $\varepsilon^2 - \varepsilon - 1 = 0$ rezulta

$$\varepsilon^2 = \varepsilon + 1 \quad \textbf{(1)},$$

de unde primim $\varepsilon^3 = \varepsilon^2.\varepsilon = (\varepsilon + 1).\varepsilon = \varepsilon^2 + \varepsilon = \varepsilon + 1 + \varepsilon = 2\varepsilon + 1.$

(ii) Din **(1)** **,** prin inmultire cu $\frac{1}{\varepsilon}$, primim $\varepsilon = 1 + \frac{1}{\varepsilon}$, i.e.

$$\varepsilon - 1 = \frac{1}{\varepsilon} \qquad \mathbf{(2)}$$

Astfel, ***detA*** $=-(\varepsilon - 1)^2 - \frac{1}{\varepsilon} \xRightarrow{(\mathbf{2})\&(\mathbf{1})}$ ***detA*** $= -\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} = -\frac{\varepsilon+1}{\varepsilon^2} = -1.$

(iii) Ecuatia ***(C-H)*** asociata matricei A este:

$A^2 - 0.A + (-1)I_2 = 0_2$, i.e. $A^2 = I_2$.

Astfel, $A^{2019} = A^{2018}.A = (A^2)^{1009}.A = (I_2)^{1009}.A = I_2.A = A$.

4. *SISTEME DE ECUATII LINIARE*

Se considera un sistem liniar de ***m*** ecuatii cu ***n*** necunoscute. Fie $\boldsymbol{A} \in \mathcal{M}_{m,n}(\mathbb{C})$ matricea sistemului si $\overline{\boldsymbol{A}} \in \mathcal{M}_{m,n+1}(\mathbb{C})$ matricea sa extinsa.

Notiuni utilizate

Sistem compatibil $\overset{def}{\Longleftrightarrow}$ ***are cel putin o solutie***

Sistem compatibil determinat $\overset{def}{\Longleftrightarrow}$ ***are exact o solutie*** $\Longleftrightarrow$ ***are solutie unica***

Sistem incompatibil $\overset{def}{\Longleftrightarrow}$ ***nu are nici o solutie***

Sistem compatibil nedeterminat $\overset{def}{\Longleftrightarrow}$ ***are o infinitate de solutii***

OBS. Daca sistemul liniar are coeficientii intr-un corp comutativ ***K,*** de exemplu $\boldsymbol{K} = \mathbb{Z}_p$, atunci sistemul este ***compatibil nedeterminat*** daca ***are cel putin doua solutii.***

4.1. Sisteme liniare de n ecuatii cu n necunoscute

In acest caz, $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{C})$.

(i) Sistemul este ***compatibil determinat*** $\Longleftrightarrow$

$\Longleftrightarrow$ ***det(A)≠ 0*** $\Longleftrightarrow$ ***rang(A) = rang($\bar{A}$) = n***

(ii) Sistemul este ***compatibil nedeterminat*** $\Longleftrightarrow$

$\Longleftrightarrow$ ***rang(A)= rang($\bar{A}$) < n*** $\Longleftrightarrow$

$\Longleftrightarrow$***det(A)= 0 si rang(A)= rang($\bar{A}$)***

(iii) Sistemul este ***incompatibil*** $\Longleftrightarrow$ ***rang(A) < rang($\bar{A}$)***

Obs. ***rang(A) < rang($\bar{A}$)*** $\Longrightarrow$ ***det(A)= 0***

4.2. Sisteme liniare si omogene de n ecuatii cu n necunoscute

Fie $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{C})$ matricea sistemului **.**

(i) Sistemul este ***compatibil determinat (adica are numai solutia nula sau banala)*** $\Longleftrightarrow$ ***det(A)≠ 0***

(ii) Sistemul este ***compatibil nedeterminat (are solutii nenule)*** $\Longleftrightarrow$ ***det(A)= 0.***

4.3. Sisteme liniare de m ecuatii cu n necunoscute

(i) Sistemul este ***compatibil*** $\Leftrightarrow$ ***rang(A) = rang($\bar{A}$)*** **(Kronecker-Capelli)**

(ii) Sistemul este ***compatibil determinat*** $\Leftrightarrow$

rang(A) = rang($\bar{A}$) = n

Obs. ***rang(A) = rang($\bar{A}$)= n*** $\Longrightarrow$ ***m≥ n***

(iii) Sistemul este ***compatibil nedeterminat*** $\Leftrightarrow$

rang(A)= rang($\bar{A}$) < n

(iv) Sistemul este ***incompatibil*** $\Leftrightarrow$ ***rang(A) < rang($\bar{A}$)***

4.4 Sisteme liniare si omogene de m ecuatii cu n necunoscute

(i) Sistemul este ***compatibil determinat (adica are numai solutia nula sau banala)*** $\Leftrightarrow$ ***rang(A)=n***

(ii) ***(ii)*** Sistemul este ***compatibil nedeterminat (are solutii nenule)*** $\Leftrightarrow$ ***rang(A) < n***

Ex. 4.1 (20/2020)

Fie P, Q, R functii de grad cel mult 2 si a, b, c numere complexe date Se considera determinantii:

$$\Delta_0 = \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(b) & Q(b) & R(b) \\ P(c) & Q(c) & R(c) \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} P(1) & Q(1) & R(1) \\ P(b) & Q(b) & R(b) \\ P(c) & Q(c) & R(c) \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(1) & Q(1) & R(1) \\ P(c) & Q(c) & R(c) \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(b) & Q(b) & R(b) \\ P(1) & Q(1) & R(1) \end{vmatrix}$$

Daca $\Delta_0 = 1$, atunci $\Delta_1 + \Delta_2 + \Delta_3$ este:

A: 0; ***B***: 1; **C**: 3; ***D***: $P(0) + Q(0) + R(0)$; **E**: P(1)Q(1)R(1)

Rezolvare.

Fie $x \in \mathbb{R}$ si

$$H(x) = \begin{vmatrix} P(x) & Q(x) & R(x) \\ P(b) & Q(b) & R(b) \\ P(c) & Q(c) & R(c) \end{vmatrix} + \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(x) & Q(x) & R(x) \\ P(c) & Q(c) & R(c) \end{vmatrix} + \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(b) & Q(b) & R(b) \\ P(x) & Q(x) & R(x) \end{vmatrix}$$

Avem H(a) = $\Delta_0 + 0 + 0 = \Delta_0 = 1$; analog H(b) =1 si H(c) =1 **(1)**

Pe de alta parte, observam ca H este o functie de gradul doi,

H(x) $= mx^2 + nx + p$ **(2)**

si $\Delta_1 + \Delta_2 + \Delta_3 = H(1)$ **(3)** .

Relatiile **(1)** & **(2)** conduc la sistemul linear $\begin{cases} ma^2 + na + p = 1 \\ mb^2 + nb + p = 1 \\ mc^2 + nc + p = 1 \end{cases}$, cu necunoscutele m, n, p. Determinantul sistemului este:

$$\Delta = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -V(a,b,c) = -(b-a)(c-a)(c-b)$$

Deoarece $\Delta_0 = 1 \neq 0$, rezulta ca numerele a, b, c sunt distincte intre ele. De aici deducem ca $\Delta \neq 0$, deci sistemul de mai sus are solutie unica, pe care o determinam cu regula lui Cramer.

Astfel, $\Delta_m = 0$ $(col\ 1 = col\ 3)$, $\Delta_n = 0$ $(col\ 2 = col\ 3)$ si $\Delta_p = \Delta$, deci m=$\frac{\Delta_m}{\Delta} = 0$, n=$\frac{\Delta_n}{\Delta} = 0$ si p = $\frac{\Delta_p}{\Delta}$ = 1, de unde obtinem, via **(2)** ,

H(x) = $mx^2 + nx + p$ = 1 , $\forall x \in \mathbb{R}$.

In final, utilizand **(3)** , obtinem $\Delta_1 + \Delta_2 + \Delta_3 = H(1) = 1.$

Ex. 4.2 (Admitere 2017; 739,749,741/2020)

Se considera sistemul de ecuatii liniare

$$\begin{cases} ax + y + z = -1 \\ x + ay + z = -a \\ x + y - z = -2 \end{cases}$$, unde $a \in \mathbb{R}$.

(i) Determinantul sistemului este:

A: a^2; ***B***: $a^2 + 2a - 3$; **C**: $a^2 - 2a + 3$; ***D***: $-a^2 - 2a + 3$; **E**: $2a + 3$

(ii) Sistemul este incompatibil daca si numai daca:

A: $a = -1$; ***B***: $a = 1$; **C**: $alt\ raspuns$; ***D***: $a \in \mathbb{R}\setminus\{-3,1\}$; **E**: $a = -3$

(iii) Numarul valorilor reale ale parametrului $a \in \mathbb{R}$ pentru care sistemul admite solutii (x, y, z), cu x, y, z in progresie aritmetica, in aceasta ordine, este:

A: 0; ***B***: 3; **C**: 1; ***D***: 2; **E**: ∞

Rezolvare.

Fie $\boldsymbol{A} = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & -1 \end{pmatrix}$ matricea sistemului dat si

$\overline{\boldsymbol{A}} = \begin{pmatrix} a & 1 & 1 & -1 \\ 1 & a & 1 & a \\ 1 & 1 & -1 & -2 \end{pmatrix}$ matricea sa extinsa.

(i) Prin calcul direct, rezulta ***detA*** = $-a^2 - 2a + 3.$

(ii) Ecuatia detA = 0 are solutiile -3 si 1.

Daca $\boldsymbol{a \in \mathbb{R} \setminus \{-3, 1\}}$, atunci sistemul este compatibil determinat.

Daca $\boldsymbol{a = 1}$, atunci $\boldsymbol{rang\ A = 2}$, alegand Δ_p= $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$, la intersectia liniilor 2 si 3 si a coloanelor 2 si 3. Mai departe,

Δ_c= $\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & -2 \end{vmatrix} = 0$ (liniile 2 si 3 sunt egale). Astfel, $\boldsymbol{rang\ A\ = rang\ \overline{A} = 2}$, deci sistemul este compatibil simplu nedeterminat.

Daca $\boldsymbol{a = -3,}$ atunci $\boldsymbol{rang\ A = 2,}$ luand acelasi Δ_p ca la situatia a = 1. In acest caz, Δ_c= $\begin{vmatrix} 1 & 1 & -1 \\ -3 & 1 & 3 \\ 1 & -1 & -2 \end{vmatrix} = -4 \neq 0$, deci

$\boldsymbol{rang\ \overline{A} = 3 \neq rang\ A}$ **.**

Asadar, $\boldsymbol{a = -3}$ este singura valoare a lui a pentru care sistemul dat este incompatibil.

(iii) Conditia de progresie aritmetica $y = \frac{x+z}{2} \Leftrightarrow \boldsymbol{x} - 2y + z + 0,$ impreuna cu sistemul dat, conduce la sistemul liniar de 4 ecuatii cu 3 necunoscute x, y, z

$$\begin{cases} ax + y + z = -1 \\ x + ay + z = -\text{a} \\ x + y - z = -2 \\ \boldsymbol{x} - 2y + z + 0 \end{cases}$$

unde $\boldsymbol{a}$ este un parametru real.

Fie $\boldsymbol{\mathcal{A}} = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$ matricea sistemului si

$\bar{\boldsymbol{\mathcal{A}}} = \begin{pmatrix} a & 1 & 1 & -1 \\ 1 & a & 1 & -a \\ 1 & 1 & -1 & -2 \\ 1 & -2 & 1 & 0 \end{pmatrix}$ matricea sa extinsa.

Observam ca $\boldsymbol{\mathcal{A}} \in \mathcal{M}_{4,3}(\mathbb{R})$, deci

$$\boldsymbol{rang(\mathcal{A}) \leq 3} \quad \textcolor{red}{(\mathbf{1})}\ .$$

Pe de alta parte, conditia de compatibilitate a sistemului (impusa de cerintele problemei), anume

$$\boldsymbol{rang(\mathcal{A}) = rang(\bar{\mathcal{A}})} \quad \textcolor{red}{(\mathbf{2})}\ ,$$

impreuna cu relatia $\textcolor{red}{(\mathbf{1})}$, conduce la egalitatea

$$\boldsymbol{det(\bar{\mathcal{A}}) = 0} \quad \textcolor{red}{(\mathbf{3})}\ .$$

Intr-adevar, daca $\det(\bar{\mathcal{A}}) \neq 0$, atunci $\text{rang}(\bar{\mathcal{A}}) = 4$, ceea ce, alaturi de relatiile $\textcolor{red}{(\mathbf{2})}$ si $\textcolor{red}{(\mathbf{1})}$, furnizeaza contradictia

$\boldsymbol{rang(\mathcal{A}) = 4 \leq 3}$.

Sa calculam $\det(\bar{\mathcal{A}})$. Scazand linia 1 din liniile 2 si 4, apoi adunand linia 1 la linia 3, obtinem

$$\boldsymbol{det(\bar{\mathcal{A}})} = \begin{vmatrix} a & 1 & 1 & -1 \\ 1-a & a-1 & 0 & 1-a \\ 1+a & 2 & 0 & -3 \\ 1-a & -3 & 0 & 1 \end{vmatrix} =$$

$$(-1)^{1+3}.1. \begin{vmatrix} 1-a & a-1 & 1-a \\ 1+a & 2 & -3 \\ 1-a & -3 & 1 \end{vmatrix} = (1-a) \begin{vmatrix} 1 & -1 & 1 \\ 1+a & 2 & -3 \\ 1-a & -3 & 1 \end{vmatrix}$$

Adunand coloana 3 la coloana 2 si scazand-o din coloana 1, primim:

$$\boldsymbol{det(\bar{\mathcal{A}})} = (1-a) \begin{vmatrix} 0 & 0 & 1 \\ 4+a & -1 & -3 \\ -a & -2 & 1 \end{vmatrix} = (a-1)(8+3a).$$

Astfel , ecuatia **(3)** are doua solutii, $a = 1$ si $a = -\frac{8}{3}$.

Prof. dr. Alexandru Mitrea