Primitive

Note Title 4/10/2021

Definiția. Fie $f: I \to \mathbb{R}$, I interval. Funcția f admite primitive pe I dacă există $F: I \to \mathbb{R}$ cu prop: i) F este derivabilă pe I;
ii) $F'(x) = f(x)$, $\forall x \in I$.

Metoda de integrare prin părți

$$\int f(x) \cdot g'(x)\,dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx, \text{ unde } f, g \in C^1(I)$$

Problema 1. i) Să se calculeze $\int \ln(x+\sqrt{x^2+1})\,dx$

ii) Să se calculeze $\int_{-1}^{1} \ln(x+\sqrt{x^2+1})\,dx$ (simulare 2018)

Rezolvare: i) $\int \ln(x+\sqrt{x^2+1})\,dx$

$f(x) = \ln(x+\sqrt{x^2+1})$ $\qquad g'(x) = 1$

$f'(x) = \dfrac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \dfrac{x}{\sqrt{1+x^2}}\right) = \dfrac{1}{\sqrt{x^2+1}}$ $\qquad g(x) = x$

$$\int \ln(x+\sqrt{x^2+1})\,dx = x\ln(x+\sqrt{x^2+1}) - \int x\cdot\frac{1}{\sqrt{x^2+1}}\,dx =$$

$$= x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + \mathcal{C}$$

ii) $F(x) = x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1}$

$$\int_{-1}^{1} \ln(x+\sqrt{x^2+1})\,dx = F(x)\Big|_{-1}^{1} = F(1) - F(-1) = \ln(1+\sqrt{2}) + \ln(-1+\sqrt{2}) =$$

$$= \ln(1+\sqrt{2})(-1+\sqrt{2}) = \ln(2-1) = \ln 1 = \underline{0}$$

Observatie : 1. $f(x) = \ln(x+\sqrt{x^2+1})$ - f e impară ?

$$f(-x) = \ln(-x+\sqrt{x^2+1}) = \ln\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x} = \ln\frac{1}{\sqrt{x^2+1}+x} =$$

$$= -f(x)\ ,\ \forall x\in\mathbb{R}$$

$\Rightarrow f(x)$ este o funcție impară $\Rightarrow \int_{-1}^{1} f(x)\,dx = 0$

2. Similar $\int_{-a}^{a} \ln\left(x^n+\sqrt{x^{2n}+1}\right)dx=0,\quad \forall n\in\mathbb{N}$ impar, $\forall a>0$

Problema 2: i) Să se calculeze $\int x\ln(1-x)dx$, $x\in(-\infty,1)$

473 ii) Să se calculeze $\lim\limits_{a\to 1}\int_0^a x\ln(1-x)dx$, $a\in(0,1)$.

Rezolvare: i) $f(x)=\ln(1-x)$ $\qquad g'(x)=x$

$f'(x)=-\frac{1}{1-x}=\frac{1}{x-1}$ $\qquad g(x)=\frac{x^2}{2}$

$$\int x\ln(1-x)dx=\frac{x^2}{2}\ln(1-x)-\frac{1}{2}\int\frac{x^2}{x-1}dx=\frac{x^2}{2}\ln(1-x)-\frac{1}{2}\int\frac{x^2-1+1}{x-1}dx=$$

$$=\frac{x^2}{2}\ln(1-x)-\frac{1}{2}\cdot\int\left((x+1)+\frac{1}{x-1}\right)dx=$$

$$=\frac{x^2}{2}\ln(1-x)-\frac{1}{2}\cdot\frac{x^2}{2}-\frac{x}{2}-\frac{1}{2}\underbrace{\ln|x-1|}_{=\ln(1-x)}+\mathcal{C}=$$

$$= \left(\frac{x^2-1}{2}\right) \ln(1-x) - \frac{x^2}{4} - \frac{x}{2} + \mathcal{C}.$$

ii) Fie o primitivă $F(x) = \frac{x^2-1}{2}\ln(1-x) - \frac{x^2}{4} - \frac{x}{2}$

$$\int_0^a x \ln(1-x)\,dx = F(x)\Big|_0^a = F(a) - F(0) = \frac{a^2-1}{2}\ln(1-a) - \frac{a^2}{4} - \frac{a}{2}$$

$$\lim_{a\to 1}\left(\frac{(a^2-1)}{2}\ln(1-a) - \frac{a^2}{4} - \frac{a}{2}\right) = -\frac{1}{4} - \frac{1}{2} + \frac{1}{2}\lim_{a\to 1}\underbrace{(a+1)}_{\substack{a\to 1\\ \to 2}}(a-1)\ln(1-a) =$$

$$= -\frac{3}{4} + \frac{1}{2}\cdot 2\lim_{a\to 1}(a-1)\ln(1-a) =$$

$$= -\frac{3}{4} - \lim_{a\to 1}(1-a)\ln(1-a) = -\frac{3}{4} - \lim_{a\to 1}\frac{\ln(1-a)}{\frac{1}{1-a}} = -\frac{3}{4} - \lim_{a\to 1}\frac{-\frac{1}{1-a}}{\frac{1}{(1-a)^2}} =$$

$$= -\frac{3}{4} + \lim_{a\to 1}(1-a) = -\frac{3}{4}.$$

Schimbare de variabilă

Problema 3. i) Să se calculeze $\int \sqrt{\frac{x}{1+x^3}}\, dx$, $x \in (0, \infty)$

ii) Să se calculeze $I = \int_0^1 \sqrt{\frac{x}{1+x^3}}\, dx$. (Simulare 2018)

Rezolvare: F o primitivă a fc. $f(x) = \sqrt{\frac{x}{1+x^3}}$, $x \in (0, \infty)$

Vom face substituția $x^{\frac{3}{2}} = t$

$\frac{3}{2} x^{\frac{1}{2}}\, dx = dt$

$$F(t) = \frac{2}{3} \int \frac{dt}{\sqrt{1+t^2}} = \frac{2}{3} \ln\left(t + \sqrt{1+t^2}\right) + \mathcal{C}$$

Revenind la integrala inițială avem $F(x) = \frac{2}{3} \ln\left(x\sqrt{x} + \sqrt{1+x^3}\right) + \mathcal{C}$.

ii) $I = F(x)\Big|_0^1 = F(1) - F(0) = \frac{2}{3} \ln(1+\sqrt{2})$.

Problema 4. Să se calculeze $\int \frac{1}{\sqrt{x(1-x)}}\, dx$, $x \in (0,1)$

Rezolvare: M. I: Notăm $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\Rightarrow F(t) = \int \frac{2dt}{\sqrt{1-t^2}} = 2\arcsin t + \mathcal{C}$$

$$F(x) = 2\arcsin\sqrt{x} + \mathcal{C}$$

M II: $$\int \frac{1}{\sqrt{x(1-x)}}dx = \int \frac{1}{\sqrt{-x^2+x}}dx = \int \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}dx =$$

$$= \arcsin \frac{x-\frac{1}{2}}{\frac{1}{2}} + \mathcal{C}_1 = \arcsin(2x-1) + \mathcal{C}_1$$

Observație 1. Din MI și MII observăm că funcțiile $g(x) = 2\arcsin\sqrt{x}$ și $h(x) = \arcsin(2x-1)$, $x \in [0,1]$ au aceeași derivată

$g'(x) = \frac{1}{\sqrt{x(1-x)}}$, $x \in (0,1) \Rightarrow$

$\Rightarrow \exists\, a \in \mathbb{R}$ a.î. $g(x) - h(x) = a$, $\forall x \in [0,1]$

pentru $x = 0 \Rightarrow a = g(0) - h(0) = 2\arcsin 0 - \arcsin(-1) = \frac{\pi}{2}$

$\Rightarrow \quad 2\arcsin\sqrt{x} = \frac{\pi}{2} + \arcsin(2x-1) \quad , \forall\, x \in [0,1]$.

2. Să se calculeze $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}}\,dx$ (Admitere 2014)

Avem: $I = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}}\,dx = 2\arcsin\sqrt{x}\Big|_{\frac{1}{4}}^{\frac{1}{2}} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$

sau

$I = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}}\,dx = \arcsin(2x-1)\Big|_{\frac{1}{4}}^{\frac{1}{2}} = 0 - \left(\arcsin\left(-\frac{1}{2}\right)\right) = \frac{\pi}{6}$.

INTEGRALE RAȚIONALE

Problema 5. i) Să se calculeze $\int \frac{1+x^2}{1+x^2+x^4}\,dx$, $x \in [0,\infty)$

ii) Să se calculeze $\int_0^1 \frac{1+x^2}{1+x^2+x^4}\,dx$ (Simulare 2017)

Rezolvare: i) F o primitivă a funcției $f(x)=\dfrac{1+x^2}{1+x^2+x^4}$, $x\in[0,\infty)$

$$F(x)=\int\frac{1+x^2}{1+x^2+x^4}dx=\int\frac{\cancel{x^2}\left(1+\frac{1}{x^2}\right)}{\cancel{x^2}\left(\frac{1}{x^2}+1+x^2\right)}dx=\int\frac{1+\frac{1}{x^2}}{\frac{1}{x^2}+1+x^2}dx$$

Notăm $x-\dfrac{1}{x}=t$, $x>0$ $\qquad\left(x-\dfrac{1}{x}\right)^2=t^2$

$\left(1+\dfrac{1}{x^2}\right)dx=dt \qquad x^2-2+\dfrac{1}{x^2}=t^2$

$$F(t)=\int\frac{dt}{t^2+3}=\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}+C$$

$$F(x)=\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x-\frac{1}{x}}{\sqrt{3}}+C,\qquad x\in(0,\infty)$$

Pentru intervalul $[0,\infty)$ avem:

$$F(x)=\begin{cases}C_1, & x=0\\ \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x^2-1}{x\sqrt{3}}+C & x>0\end{cases}$$

Deoarece F este continuă pe $[0,\infty)$ $\Rightarrow$

$$c_1 = \lim_{x\to 0}\left(\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x^2-1}{x\sqrt{3}} + \mathcal{C}\right) = \frac{1}{\sqrt{3}}\operatorname{arctg}(-\infty) + \mathcal{C} = -\frac{\pi}{2\sqrt{3}} + \mathcal{C}$$

$$\Rightarrow \quad F(x) = \begin{cases} -\frac{\pi}{2\sqrt{3}} + \mathcal{C}, & x = 0 \\ \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x^2-1}{x\sqrt{3}} + \mathcal{C}, & x \in (0,\infty). \end{cases}$$

Observație: O primitivă a funcției $f(x)$ se poate calcula utilizând metoda clasică, astfel:

$$f(x) = \frac{x^2+1}{x^4+x^2+1} = \frac{x^2+1}{x^4+x^2+1+x^2-x^2} = \frac{x^2+1}{(x^2+1)^2-x^2}$$

$$= \frac{1}{2}\frac{(x^2+1+x)+(x^2+1-x)}{(x^2+1+x)(x^2+1-x)} = \frac{1}{2}\cdot\frac{1}{x^2+x+1} + \frac{1}{2}\frac{1}{x^2-x+1}$$

$$= \frac{1}{2} \cdot \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{2} \cdot \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\Rightarrow G(x) = \int \frac{x^2+1}{x^4+x^2+1}\,dx = \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C_2$$

$$= \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C_2$$

ii) $$I = \int_0^1 \frac{1+x^2}{1+x^2+x^4}\,dx = F(x)\Big|_0^1 = F(1) - F(0) = -\left(-\frac{\pi}{2\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}$$

sau

$$I = G(x)\Big|_0^1 = G(1) - G(0) = \frac{1}{\sqrt{3}}\left(\operatorname{arctg}\frac{1}{\sqrt{3}} + \operatorname{arctg}\sqrt{3} + \operatorname{arctg}\frac{1}{\sqrt{3}} - \operatorname{arctg}\frac{1}{\sqrt{3}}\right) =$$

$$= \frac{\pi}{2\sqrt{3}}.$$

Problema 6: Fie $f_n : \mathbb{R}_+^* \to \mathbb{R}$, $f_n(x) = \dfrac{1}{x(x^n+1)}$, $I_n = \int f_n(x)dx$.

i) Calculați I_1;

ii) Calculați I_2;

iii) Calculați I_3 (439, Ed: 2020)

iv) Calculați $\lim_{n\to\infty} n\int_1^2 f_n(x)dx$;

(498) v) Dacă $F_n : (0,\infty) \to \mathbb{R}$ este primitiva funcției f_n al cărui grafic trece prin $A(1,0)$, atunci soluția inecuației $\left|\lim_{n\to\infty} F_n(x)\right| \le 1$ este...

Rezolvare: i) $I_1 = \int \frac{1}{x(x+1)}dx = \int \frac{x+1-x}{x(x+1)}dx = \int\left(\frac{1}{x} - \frac{1}{1+x}\right)dx = \ln x - \ln(1+x) + \mathcal{C}$

$= \ln\frac{x}{x+1} + \mathcal{C}$

ii) $I_2 = \int \frac{1}{x(x^2+1)}dx = \int \frac{x}{x^2(x^2+1)}dx$

F_2 o primitivă, notăm $x^2 = t$

$2x\,dx = dt$

$F_2(t) = \frac{1}{2}\int \frac{1}{t(t+1)}\,dt = \frac{1}{2}\ln\frac{t}{t+1} + C$; $F_2(x) = \frac{1}{2}\ln\frac{x^2}{x^2+1} + C$

ne intoarcem la integrala initiala: $I_2 = \int \frac{dx}{x(x^2+1)} = \frac{1}{2}\ln\frac{x^2}{x^2+1} + C$.

iii) $I_3 = \int \frac{1}{x(x^3+1)}\,dx = \int \frac{x^2}{x^3(x^3+1)}\,dx$

Notam $x^3 = t \Rightarrow 3x^2dx = dt$

$F_3(t) = \frac{1}{3}\int \frac{1}{t(t+1)}\,dt = \frac{1}{3}\ln\frac{t}{t+1} + C$

$I_3 = \frac{1}{3}\ln\frac{x^3}{x^3+1} + C$

iv) $F_n(x)$ primitiva a $f_n(x) = \frac{1}{x(x^n+1)}$

$F_n(x) = \int \frac{1}{x(x^n+1)}\,dx = \int \frac{x^{n-1}}{x^n(x^n+1)}\,dx$

notăm $x^n = t$

$n x^{n-1} dx = dt$

$$F_n(t) = \frac{1}{n}\int \frac{dt}{t(t+1)} = \frac{1}{n}\ln\frac{t}{t+1} + C$$

Așadar $F_n(x) = \dfrac{1}{n}\ln\dfrac{x^n}{x^n+1} + C.$

(O altă metodă cu substituția $x = \frac{1}{t}$).

Calculăm $\displaystyle\lim_{n\to\infty} n\int_1^2 f_n(x)\,dx = \lim_{n\to\infty} n\cdot\frac{1}{n}\ln\frac{x^n}{x^n+1}\Big|_1^2 =$

$$= \lim_{n\to\infty}\left(\ln\frac{2^n}{2^n+1} - \ln\frac{1}{2}\right) = -\ln\frac{1}{2} = \ln 2.$$

(v) $F_n(x)$ al cărui grafic trece prin $A(1,0)$

$F_n(x) = \dfrac{1}{n}\ln\dfrac{x^n}{x^n+1} + C$

$$F_n(1)=0 \Leftrightarrow \frac{1}{n}\ln\frac{1}{2}+\mathcal{C}=0 \Rightarrow \mathcal{C}=\frac{\ln 2}{n}$$

$$\Rightarrow F_n(x)=\frac{1}{n}\ln\frac{x^n}{x^n+1}+\frac{1}{n}\ln 2=\frac{1}{n}\ln\frac{2x^n}{x^n+1}, \quad x\in(0,\infty)$$

Dacă $x=1 \Rightarrow \lim\limits_{n\to\infty}\frac{1}{n}\ln 2=0$

Dacă $x>1 \Rightarrow \lim\limits_{n\to\infty}\frac{1}{n}\ln\frac{2x^n}{x^n+1}=\lim\limits_{n\to\infty}\frac{1}{n}\cdot\ln\frac{2}{1+\frac{1}{x^n}}=0<1$

Dacă $x<1$
$x\in(0,1)$ $\Rightarrow \lim\limits_{n\to\infty}\frac{1}{n}\ln\frac{2x^n}{x^n+1}=\lim\limits_{n\to\infty}\left(\frac{\ln 2}{n}+\frac{1}{n}\ln x^n-\frac{1}{n}\ln(x^n+1)\right)=$

$$=0+\ln x-0=\ln x$$

$$\left|\lim_{n\to\infty}F_n(x)\right|\le 1 \Leftrightarrow |\ln x|\le 1,\ x\in(0,1) \Leftrightarrow -\ln x\le 1 \Leftrightarrow$$

$$\Leftrightarrow \ln x\ge -1 \Leftrightarrow x\ge\frac{1}{e}$$

$\Rightarrow$ inecuația $\left|\lim\limits_{n\to\infty} F_n(x)\right| \leq 1$ are ca soluție $[\frac{1}{e}, \infty)$.

Problema 7: i) Să se calculeze $\int \frac{x-x^2}{(x^2+1)(x^3+1)}\,dx$, $x \in [0,\infty)$.

ii) Să se calculeze $L = \lim\limits_{n\to\infty} \int_0^n \frac{x-x^2}{(x^2+1)(x^3+1)}\,dx$ (Admitere 2019)

Rezolvare: i) $\frac{x-x^2}{(x^2+1)(x^3+1)} = \frac{x-x^2+x^4-x^4}{(x^2+1)(x^3+1)} = \frac{(x+x^4)-(x^2+x^4)}{(x^2+1)(x^3+1)} =$

$$= \frac{x(1+x^3)-x^2(1+x^2)}{(x^2+1)(x^3+1)} = \frac{x}{x^2+1} - \frac{x^2}{x^3+1}$$

Deci $F(x) = \int \frac{x}{x^2+1}\,dx - \int \frac{x^2}{x^3+1}\,dx = \frac{1}{2}\ln(x^2+1) - \frac{1}{3}\ln(x^3+1) + \mathcal{C}$

$$= \frac{1}{6}\ln\frac{(x^2+1)^3}{(x^3+1)^2} + \mathcal{C}$$

ii) $L = \lim_{n\to\infty}\left(F(n)-F(0)\right) = \lim_{n\to\infty} \frac{1}{6}\ln\frac{(n^2+1)^3}{(n^3+1)^2} =$

$$= \frac{1}{6}\lim_{n\to\infty} \ln \frac{n^6+3n^4+3n^2+1}{n^6+2n^3+1}$$

$$= \frac{1}{6}\ln 1 = 0.$$

Observație: O generalizare a integralei de la i) este următoarea:

Fie $a, b \geq 1$ și $x \in [0,\infty)$, avem

$$\int \frac{x^{b-1}-x^{a-1}}{(x^a+1)(x^b+1)}\,dx = \int \frac{x^{b-1}+x^{a+b-1}-x^{a-1}-x^{a+b-1}}{(x^a+1)(x^b+1)}\,dx =$$

$$= \int \frac{x^{b-1}(1+x^a)-x^{a-1}(1+x^b)}{(1+x^a)(1+x^b)}\,dx =$$

$$= \int \frac{x^{b-1}}{x^b+1}\,dx - \int \frac{x^{a-1}}{x^a+1}\,dx =$$

$$= \frac{1}{b}\ln(x^b+1) - \frac{1}{a}\ln(x^a+1) + \mathcal{C} =$$

$$= \frac{1}{ab}\ln\frac{(x^b+1)^a}{(x^a+1)^b} + \mathcal{C}.$$

Integrale trigonometrice

Problema 8: Să se calculeze $\int_0^{4\pi} \frac{dx}{5+4\cos x}$, (475, ED 2020)

Rezolvare: $\int_0^{4\pi} \frac{dx}{5+4\cos x} = \int_0^{2\pi} \frac{dx}{5+4\cos x} + \int_{2\pi}^{4\pi} \frac{dx}{5+4\cos x}\,dx = \int_0^{2\pi} \frac{dx}{5+4\cos x} + \int_{2\pi}^{2\pi+2\pi} \frac{dx}{5+4\cos x}$

$x = 2\pi$

$T = 2\pi$

$$\int_x^{x+T} f(x)\,dx = \int_0^T f(x)\,dx$$ unde f este o funcție periodică de perioadă T

$$= 2\int_0^{2\pi} \frac{dx}{5+4\cos x} = 2\int_{-\pi}^{\pi} \frac{dx}{5+4\cos x} \underset{\uparrow}{=} 2\cdot 2\int_0^{\pi} \frac{dx}{5+4\cos x} = 4\int_0^{\pi}\frac{dx}{5+4\cos x}$$

$x = -\frac{T}{2} = -\pi$

fc pară

Fie $F(x)$ o primitivă a funcției $f(x) = \frac{1}{5+4\cos x}$

$\operatorname{tg}\frac{x}{2} = t \Rightarrow x = 2\operatorname{arctg} t$ $\qquad \cos x = \frac{1-t^2}{1+t^2}$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\Rightarrow F(t) = \int \frac{1}{5+4\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{t^2+9}\,dt = \frac{2}{3}\operatorname{arctg}\frac{t}{3} + C$$

$$F(x) = \frac{2}{3}\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}}{3} + C, \quad x \in [0,\pi)$$

$$F(x) = \begin{cases} \mathcal{C}_1, & x = \pi \\ \frac{2}{3}\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}}{3} + \mathcal{C}, & x \in [0, \pi) \end{cases}$$

$F(x)$ este continuă pe $[0,\pi]$ $\longrightarrow$

$$\mathcal{C}_1 = \lim_{x\to\pi}\left(\frac{2}{3}\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}}{3} + \mathcal{C}\right) = \frac{2}{3}\cdot\frac{\pi}{2} + \mathcal{C} = \frac{\pi}{3} + \mathcal{C}$$

Așadar $$F(x) = \begin{cases} \frac{\pi}{3} + \mathcal{C}, & x = \pi \\ \frac{2}{3}\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}}{3} + \mathcal{C}, & x \in [0, \pi) \end{cases}$$

Revenind la integrala inițială $\displaystyle\int_0^{4\pi}\frac{dx}{5+4\cos x} = 4\int_0^{\pi}\frac{dx}{5+4\cos x} = 4F(x)\Big|_0^{\pi} = 4(F(\pi) - F(0)) =$

$$= 4\left(\frac{\pi}{3} - 0\right) = \frac{4\pi}{3}.$$

Alte tipuri de integrale.

443: $I = \int_{-2}^{0} \frac{x}{\sqrt{e^x + (x+2)^2}}\, dx = \int_{-2}^{0} \frac{x}{e^{\frac{x}{2}}\sqrt{1 + (x+2)^2 e^{-x}}}\, dx =$

$$= \int_{-2}^{0} \frac{x \cdot e^{-\frac{x}{2}}}{\sqrt{1 + \left[(x+2)e^{-\frac{x}{2}}\right]^2}}\, dx$$

Notăm $(x+2)e^{-\frac{x}{2}} = t$

$\left[e^{-\frac{x}{2}} + (x+2)e^{-\frac{x}{2}} \cdot \left(-\frac{1}{2}\right)\right] dx = dt$ $\qquad x = -2 \Rightarrow t = 0$

$e^{-\frac{x}{2}} \cdot \left(1 - \frac{x}{2} - 1\right) dx = dt$ $\qquad x = 0 \Rightarrow t = 2$

$-\frac{x}{2} e^{-\frac{x}{2}}\, dx = dt \Rightarrow +x e^{-\frac{x}{2}} = -2dt$

$$\Rightarrow I = \int_0^2 \frac{-2dt}{\sqrt{1+t^2}} = -2\ln\left(t + \sqrt{1+t^2}\right)\Big|_0^2 = -2\ln(2+\sqrt{5}).$$

440. $\int \frac{e^x(x^2-2x+1)}{(x^2+1)^2}dx, \ x\in\mathbb{R}$

Avem $\int \frac{e^x(x^2-2x+1)}{(x^2+1)^2}dx = \int\left(\frac{e^x(x^2+1)}{(x^2+1)^2} - \frac{2x}{(x^2+1)^2}e^x\right)dx$

$$= \int \frac{e^x}{x^2+1}dx + \int\left(\frac{1}{x^2+1}\right)' e^x\, dx =$$

$$= \int \frac{e^x}{x^2+1}dx + \frac{e^x}{1+x^2} - \int \frac{e^x}{1+x^2}dx = \frac{e^x}{1+x^2} + \mathcal{C}$$

442. $\int \frac{e^x+\cos x}{e^x+\cos x+\sin x}dx \ , \ x\in[0,\infty)$

Rezolvare: $f(x) = e^x + \cos x + \sin x$

$$f'(x) = e^x - \sin x + \cos x$$

Observăm $f(x) + f'(x) = 2(e^x + \cos x) \Rightarrow e^x + \cos x = \frac{1}{2}(f(x) + f'(x))$

$\Rightarrow$ Avem $\displaystyle\int \frac{e^x + \cos x}{e^x + \cos x + \sin x}\,dx = \frac{1}{2}\int \frac{f(x) + f'(x)}{f(x)}\,dx =$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{f'(x)}{f(x)}\,dx = \frac{1}{2}x + \frac{1}{2}\ln(e^x + \cos x + \sin x) + \mathcal{C}.$$