Ecuaţii şi inecuaţii I

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30. Mulţimea soluţiilor inecuaţiei

$$\log_{\frac{1}{3}}(\log_3 x) \geq 1 \; este : ...$$

Soluţie. Condiţii:

$$\left.\begin{array}{c} x > 0 \\ \log_3 x > 0 \Rightarrow \log_3 x > \log_3 1 \Rightarrow x > 1 \end{array}\right\} \Rightarrow x > 1 \tag{1}$$

$$\log_{\frac{1}{3}}(\log_3 x) \geq 1 \Rightarrow \log_{\frac{1}{3}}(\log_3 x) \geq \log_{\frac{1}{3}} \frac{1}{3} \Rightarrow \log_3 x \leq \frac{1}{3} = \log_3 3^{\frac{1}{3}}$$

$$\Rightarrow x \leq 3^{\frac{1}{3}} = \sqrt[3]{3} \tag{2}$$

Din (1) şi (2) $\Rightarrow x \in (1, \sqrt[3]{3}]$.

Răspuns corect: C.

134. Mulţimea soluţiilor inecuaţiei

$$\log_x(1+x) + \log_{x^2}(1+x) + \log_{x^4}(1+x) \geq \frac{7}{4} \; este : ...$$

Soluţie.

Condiţii:

$$\begin{cases} 1+x > 0 \\ x, x^2, x^4 > 0 \\ x, x^2, x^4 \neq 1 \end{cases} \Rightarrow x \in (0,1) \cup (1, \infty)$$

Avem

$$\log_{x^2}(1+x) = \frac{\log_x(1+x)}{\log_x x^2} = \frac{\log_x(1+x)}{2}$$

$$\log_{x^4}(1+x) = \frac{\log_x(1+x)}{\log_x x^4} = \frac{\log_x(1+x)}{4}$$

Rezultă

$$\log_x(1+x) + \frac{\log_x(1+x)}{2} + \frac{\log_x(1+x)}{4} \geq \frac{7}{4}$$

$$\Rightarrow \log_x(1+x)\left(1 + \frac{1}{2} + \frac{1}{4}\right) \geq \frac{7}{4}$$

$$\Rightarrow \log_x(1+x) \geq 1$$

Cazuri:

I. $x \in (0,1)$

$$\log_x(1+x) \geq 1 \Rightarrow \log_x(1+x) \geq \log_x x \Rightarrow 1+x \leq x \Rightarrow 1 \leq 0(F) \Rightarrow S_1 = \emptyset$$

II. $x \in (1, \infty)$

$$\log_x(1+x) \geq 1 \Rightarrow \log_x(1+x) \geq \log_x x \Rightarrow 1+x \geq x \Rightarrow 1 \geq 0(A)$$

$$\Rightarrow S_2 = (1, \infty)$$

Deci

$$S = S_1 \cup S_2 = (1, \infty)$$

Răspuns corect: B.

97. Mulţimea soluţiilor reale ale ecuaţiei

$$\sqrt{x+2-4\sqrt{x-2}} + \sqrt{x+7-6\sqrt{x-2}} = 1 \ este : ...$$

Soluţie. Condiţii:

$$\begin{cases} x-2 \geq 0 \Rightarrow x \geq 2 \\ x+2-4\sqrt{x-2} \geq 0 \Rightarrow x-2-4\sqrt{x-2}+4 = \left(\sqrt{x-2}-2\right)^2 \geq 0(A) \\ x+7-6\sqrt{x-2} \geq 0 \Rightarrow x-2-6\sqrt{x-2}+9 = \left(\sqrt{x-2}-3\right)^2 \geq 0(A) \end{cases}$$

$$\sqrt{x+2-4\sqrt{x-2}} + \sqrt{x+7-6\sqrt{x-2}} = 1$$

$$\Leftrightarrow \sqrt{\left(\sqrt{x-2}-2\right)^2} + \sqrt{\left(\sqrt{x-2}-3\right)^2} = 1$$

$$\Leftrightarrow \left|\sqrt{x-2}-2\right| + \left|\sqrt{x-2}-3\right| = 1$$

Notăm

$$\sqrt{x-2} = y \geq 0 \Rightarrow |y-2| + |y-3| = 1$$

y	0 2 3 +∞
$y-2$	− − − − 0 + + + + + + + + + +
$y-3$	− − − − − − − 0 + + + + + + + +

Cazuri:
I. $y \in [0,2) \Rightarrow 2-y+3-y = 1 \Rightarrow -2y = -4 \Rightarrow y = 2 \notin [0,2)$
II. $y \in [2,3) \Rightarrow y-2+3-y = 1 \Rightarrow 1 = 1(A) \Rightarrow y \in [2,3)$
III. $y \in [3,\infty) \Rightarrow y-2+y-3 = 1 \Rightarrow 2y = 6 \Rightarrow y = 3 \in [3,\infty)$
I+II+III$\Rightarrow$ $y \in [2,3] \Rightarrow 2 \leq y \leq 3 \Rightarrow 2 \leq \sqrt{x-2} \leq 3 \Rightarrow 4 \leq x-2 \leq 9 \Rightarrow 6 \leq x \leq 11$
Răspuns corect: D.

112. Mulţimea soluţiilor reale ale ecuaţiei

$$\sqrt{x^2-4x+4} + \sqrt{x^2+4x+4} = 2x \ este : ...$$

Soluţie. Condiţii:

$$\begin{cases} x^2-4x+4 \geq 0 \Rightarrow (x-2)^2 \geq 0(A) \\ x^2+4x+4 \geq 0 \Rightarrow (x+2)^2 \geq 0(A) \\ \sqrt{x^2-4x+4} + \sqrt{x^2+4x+4} \geq 0 \Rightarrow 2x \geq 0 \Rightarrow x \geq 0 \end{cases}$$

$$\sqrt{x^2-4x+4}+\sqrt{x^2+4x+4}=2x$$
$$\Leftrightarrow \sqrt{(x-2)^2}+\sqrt{(x+2)^2}=2x$$
$$\Leftrightarrow |x-2|+|x+2|=2x$$

x	$0 \qquad 2 \qquad\qquad +\infty$
$x-2$	$- - - - 0 + + + + + + + + +$
$x+2$	$+ + + + + + + + + + + + + + +$

Cazuri:
I. $x \in [0,2) \Rightarrow -x+2+x+2=2x \Rightarrow 4=2x \Rightarrow x=2 \notin [0,2) \Rightarrow S_1=\emptyset$
II. $x \in [2,\infty) \Rightarrow x-2+x+2=2x \Rightarrow 2x=2x(A) \Rightarrow S_2=[2,\infty)$
$S=S_1 \cup S_2=[2,\infty)$
Răspuns corect: C.

113. Soluţia ecuaţiei

$$(3-2\sqrt{2})^x-2\left(\sqrt{2}-1\right)^x=3 \ \textit{este} : ...$$

Soluţie.

$$\left(\sqrt{2}-1\right)^2=2-2\sqrt{2}+1=3-2\sqrt{2}$$
$$\left(\sqrt{2}-1\right)^{2x}-2\left(\sqrt{2}-1\right)^x=3$$

Notăm

$$\left(\sqrt{2}-1\right)^x=t>0$$
$$t^2-2t=3 \Rightarrow t^2-2t-3=0 \Rightarrow t_1=3, t_2=-1$$
$$t_1=3 \Rightarrow \left(\sqrt{2}-1\right)^x=3 \Rightarrow \log_3\left(\sqrt{2}-1\right)^x=\log_3 3$$
$$\Rightarrow x\log_3\left(\sqrt{2}-1\right)=1 \Rightarrow x=\frac{1}{\log_3\left(\sqrt{2}-1\right)}$$
$$t_2=-1<0 \ \textit{nu convine}$$

Răspuns corect: E.

114. Soluţia ecuaţiei

$$\left(\frac{1}{2}\right)^x+\left(\frac{1}{6}\right)^x+\sqrt{2}\left(\frac{\sqrt{2}}{6}\right)^x=1 \ \textit{este} : ...$$

Soluţie.
Fie

$$f:\mathbb{R}\to\mathbb{R}, f(x)=\left(\frac{1}{2}\right)^x+\left(\frac{1}{6}\right)^x+\sqrt{2}\left(\frac{\sqrt{2}}{6}\right)^x-1$$

$$f'(x)=\left(\frac{1}{2}\right)^x \ln\frac{1}{2}+\left(\frac{1}{6}\right)^x \ln\frac{1}{6}+\sqrt{2}\left(\frac{\sqrt{2}}{6}\right)^x \ln\frac{\sqrt{2}}{6}<0,\ x\in\mathbb{R}$$

$\Rightarrow f$ este strict descrescătoare$\Rightarrow f$ este injectivă.

Avem

$$\left.\begin{array}{c} f(1)=\frac{1}{2}+\frac{1}{6}+\sqrt{2}\left(\frac{\sqrt{2}}{6}\right)-1=0 \\ f \text{ injectivă} \end{array}\right\}\Rightarrow$$

x=1 este singura soluţie a ecuaţiei.

Răspuns corect: B.

115. Ecuaţia

$$\left(5+\sqrt{24}\right)^{\sqrt{x+1}}+\left(5-\sqrt{24}\right)^{\sqrt{x+1}}=98$$

are mulţimea soluţiilor: ...

Soluţie.

Condiţie: $x+1\geq 0\Rightarrow x\geq -1$

Avem

$$\left(5+\sqrt{24}\right)\left(5-\sqrt{24}\right)=25-24=1\Rightarrow 5-\sqrt{24}=\frac{1}{5+\sqrt{24}}$$

Notăm

$$\left(5+\sqrt{24}\right)^{\sqrt{x+1}}=t>0$$

$$\Rightarrow t+\frac{1}{t}=98\Rightarrow t^2-98t+1=0$$

$$\Delta=98^2-4=(98-2)(98+2)=96\cdot 100=6\cdot 4^2\cdot 10^2$$

$$t_{1,2}=\frac{98\pm 40\sqrt{6}}{2}=49\pm 20\sqrt{6}=\left(5\pm\sqrt{24}\right)^2$$

$$49\pm 20\sqrt{6}=24+25\pm 2\cdot 5\sqrt{24}=\left(5\pm\sqrt{24}\right)^2$$

Cazuri:

I

$$t=t_1\Rightarrow\left(5+\sqrt{24}\right)^{\sqrt{x+1}}=\left(5+\sqrt{24}\right)^2\Rightarrow\sqrt{x+1}=2\Rightarrow x+1=4$$

$$\Rightarrow x=3$$

II

$$t=t_2\Rightarrow\left(5+\sqrt{24}\right)^{\sqrt{x+1}}=\left(5-\sqrt{24}\right)^2$$

$$\Rightarrow\left(5+\sqrt{24}\right)^{\sqrt{x+1}}=\left(5+\sqrt{24}\right)^{-2}\Rightarrow\sqrt{x+1}=-2\ fals$$

Răspuns corect: A.

123. Mulţimea soluţiilor ecuaţiei

$$\log_3 x^2 - 2\log_{-x} 9 = 2 \ \textit{este} \ ...$$

Soluţie
Condiţii:

$$\begin{cases} -x > 0 \Rightarrow x < 0 \\ -x \neq 1 \Rightarrow x \neq -1 \\ x^2 > 0 \Rightarrow x \neq 0 \end{cases} \Rightarrow x \in (-\infty, -1) \cup (-1, 0)$$

Avem

$$\log_3 x^2 = \log_3(-x)^2 = 2\log_3(-x)$$

$$\log_{-x} 9 = \frac{\log_3 9}{\log_3(-x)} = \frac{2}{\log_3(-x)}$$

$$\log_a b = \frac{\log_c b}{\log_c a}, \ a, b, c > 0, \ a, c \neq 1$$

Notăm

$$\log_3(-x) = t \neq 0$$

$$2t - \frac{4}{t} = 2 \Rightarrow t - \frac{2}{t} = 1 \Rightarrow t^2 - t - 2 = 0 \Rightarrow t_1 = 2, t_2 = -1$$

$$t = t_1 \Rightarrow \log_3(-x) = 2 \Rightarrow \log_3(-x) = \log_3 9 \Rightarrow -x = 9 \Rightarrow x = -9$$

$$t = t_2 \Rightarrow \log_3(-x) = -1 \Rightarrow \log_3(-x) = \log_3 \frac{1}{3} \Rightarrow -x = \frac{1}{3} \Rightarrow x = -\frac{1}{3}$$

Răspuns corect: E.

129. Valoarea expresiei

$$\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2} \ \textit{este} : ...$$

Soluţie.
Folosim formula

$$(x-y)^3 = x^3 - y^3 - 3xy(x-y), \ x, y \in \mathbb{R}$$

Notăm

$$\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2} = t$$

$$\Rightarrow t^3 = \sqrt{5} + 2 - (\sqrt{5} - 2) - 3\sqrt[3]{\sqrt{5}+2}\sqrt[3]{\sqrt{5}-2} \cdot t$$

$$\Rightarrow t^3 = 4 - 3t \Rightarrow t^3 + 3t - 4 = 0 \Rightarrow (t-1)(t^2 + t + 4) = 0$$

Deoarece

$$t^2 + t + 4 \neq 0 (\Delta < 0) \Rightarrow t - 1 = 0 \Rightarrow t = 1.$$

Deci

$$\sqrt[3]{\sqrt{5}+2}-\sqrt[3]{\sqrt{5}-2}=1$$

Răspuns corect: A.

131. Ştiind că a este rădăcina reală a ecuaţiei $x^3+x+1=0$ să se calculeze

$$\sqrt[3]{(3a^2-2a+2)(3a^2+2a)}+a^2.$$

Soluţie.
Avem

$$a^3+a+1=0$$

$$\begin{aligned} 3a^2-2a+2 &= 3a^2-2a+2-0=3a^2-2a+2-(a^3+a+1)\\ &= 1-3a+3a^2-a^3=(1-a)^3 \end{aligned}$$

$$\begin{aligned} 3a^2+2a &= 3a^2+2a+0=3a^2+2a+a^3+a+1\\ &= a^3+3a^2+3a+1=(1+a)^3 \end{aligned}$$

$$\begin{aligned} \sqrt[3]{(3a^2-2a+2)(3a^2+2a)}+a^2 &= \sqrt[3]{(1-a)^3(1+a)^3}+a^2\\ &= (1-a)(1+a)+a^2=1-a^2+a^2=1 \end{aligned}$$

Răspuns corect: B.

101. Câte soluţii întregi are ecuaţia

$$8(4^x+4^{-x})-54(2^x+2^{-x})+101=0?$$

Soluţie. Notăm

$$2^x=t>0.$$

Se obţine

$$8\left(t^2+\frac{1}{t^2}\right)-54\left(t+\frac{1}{t}\right)+101=0$$

Notăm

$$t+\frac{1}{t}=u>0\Rightarrow u^2=t^2+\frac{1}{t^2}+2\Rightarrow t^2+\frac{1}{t^2}=u^2-2$$

Rezultă

$$8(u^2-2)-54u+101=0\Rightarrow 8u^2-54u+85=0$$

$$\Delta=(-54)^2-4\cdot 8\cdot 85=2916-2720=196$$

$$u_{1,2}=\frac{54\pm 14}{16}\Rightarrow u_1=\frac{68}{16}=\frac{17}{4};u_2=\frac{40}{16}=\frac{5}{2}$$

Cazuri:
I
$$u_1 = \frac{17}{4} \Rightarrow t + \frac{1}{t} = \frac{17}{4} \Rightarrow 4t^2 - 17t + 4 = 0$$
$$\Delta = (-17)^2 - 4 \cdot 4 \cdot 4 = 289 - 64 = 225$$
$$t_{1,2} = \frac{17 \pm 15}{8}$$
$$t = 4 \Rightarrow 2^x = 4 \Rightarrow x = 2$$
$$t = \frac{1}{4} \Rightarrow 2^x = 2^{-2} \Rightarrow x = -2$$
II
$$u_2 = \frac{5}{2} \Rightarrow t + \frac{1}{t} = \frac{5}{2} \Rightarrow 2t^2 - 5t + 2 = 0$$
$$\Delta = (-5)^2 - 4 \cdot 2 \cdot 2 = 9$$
$$t_{1,2} = \frac{5 \pm 3}{4}$$
$$t = 2 \Rightarrow 2^x = 2 \Rightarrow x = 1$$
$$t = \frac{1}{2} \Rightarrow 2^x = 2^{-1} \Rightarrow x = -1$$
Mulţimea soluţiilor este
$$S = \{-2, -1, 1, 2\} \Rightarrow 4 \text{ soluţii întregi}$$
Răspuns corect: B.

216. Multimea valorilor $a \in \mathbb{R}$ pentru care $\log_a(x^2 + 4) \geq 2, \forall x \in \mathbb{R}$ este ...
Soluţie. Condiţii: $a > 0$, $a \neq 1$.
Cazuri:
I $a \in (0, 1)$
$$\log_a(x^2 + 4) \geq 2 \Leftrightarrow \log_a(x^2 + 4) \geq \log_a a^2 \Leftrightarrow x^2 + 4 \leq a^2, \forall x \in \mathbb{R}$$
$$x = a \Rightarrow a^2 + 4 \leq a^2 \Rightarrow 4 \leq 0 (F)$$
II $a \in (1, \infty)$
$$\log_a(x^2 + 4) \geq 2 \Leftrightarrow \log_a(x^2 + 4) \geq \log_a a^2 \Leftrightarrow x^2 + 4 \geq a^2, \forall x \in \mathbb{R}$$
$$x^2 + 4 \geq 4, \forall x \in \mathbb{R} \Rightarrow a^2 \leq 4 \Rightarrow a \in [-2, 2], a \in (1, \infty) \Rightarrow a \in (1, 2]$$
Răspuns corect: A.

25.

$$\log_{1+x}(2x^3 + 2x^2 - 3x + 1) = 3$$

Soluţie. Condiţii:

$$\begin{cases} 1 + x > 0 \\ 1 + x \neq 1 \\ 2x^3 + 2x^2 - 3x + 1 > 0 \end{cases}$$

Avem

$$\log_{1+x}(2x^3 + 2x^2 - 3x + 1) = \log_{1+x}(1 + x)^3$$

$$\Leftrightarrow 2x^3 + 2x^2 - 3x + 1 = 1 + 3x + 3x^2 + x^3$$

$$\Leftrightarrow x^3 - x^2 - 6x = 0$$

$$\Rightarrow x(x^2 - x - 6) = 0 \Rightarrow x_1 = 0, x_2 = -2, x_3 = 3$$

x_1, x_2 nu verifică condiţiile $\Rightarrow S = \{3\}$.
Răspuns corect: C.

2.

$$x(1 - \lg 5) = \lg(2^x + x - 1)$$

Soluţie. Condiţie:

$$2^x + x - 1 > 0$$

Avem

$$x - x \lg 5 = \lg(2^x + x - 1)$$

$$\Leftrightarrow \lg 10^x - \lg 5^x = \lg(2^x + x - 1)$$

$$\Leftrightarrow \lg 2^x = \lg(2^x + x - 1) \Rightarrow x = 1$$

$x = 1$ verifică condiţia $\Rightarrow S = \{1\}$.
Răspuns corect: C.