ALGEBRA LINIARA

1. **Ecuatia Cayley-Hamilton**
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3. **Sisteme de ecuatii liniare**
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1. *ECUATIA CAYLEY-HAMILTON (C-H)*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{C})$$

$$\xRightarrow{calcul} \begin{cases} A^2 - (a+d)A + (ad-bc)I_2 = 0_2 \\ A^2 - (TrA)A + \det(A)I_2 = 0_2 \end{cases}$$

Problema 1.1 (Admitere 2009)

Fie $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{Z})$.

(i) Numarul solutiior ecuatiei $X^2 = A$, $X \epsilon \mathcal{M}_2(\mathbb{R})$este:

A: 1; ***B***: 0 ; **C**: 2; ***D***: 3; **E**: ∞

(ii) Numarul solutiior ecuatiei $X^3 = A$, $X \epsilon \mathcal{M}_2(\mathbb{Z})$ este:

A: 1; ***B***: 5 ; **C**: 2; ***D***: 0; **E**: ∞

(iii) Daca $A^3 = a_3 A + b_3 I_2$, atunci perechea (a_3, b_3) este:

A: $(1,5)$; ***B***: $(\mathbf{2}1,25)$; **C**: $(27,10)$; ***D***: $(27,15)$; **E**: $(25,10)$

(iv) Daca $A^n = a_n A + b_n I_2$, atunci :

A: $a_{n+1} = 5a_n + b_n$; ***B***: $a_{n+1} = 5a_n + 2b_n$; **C**: $a_{n+1} = 3a_n + b_n$; ***D***: $a_{n+1} = -5a_n + 2b_n$; **E**: $b_{n+1} = 3a_n$

Rezolvare

(i) $X^2 = A \xRightarrow{propr} \det(X^2) = \det(A) \xRightarrow{propr} (\det X)^2 = -2$, imposibil, deoarece $X \epsilon \mathcal{M}_2(\mathbb{R})$, deci $detX \epsilon\ \mathbb{R}$ si $(\det X)^2 \geq 0$.

(ii) $X^3 = A \xRightarrow{propr} \det(X^3) = \det(A) \xRightarrow{propr} (\det X)^3 = -2$, imposibil. Intr-adevar, $X \epsilon \mathcal{M}_2(\mathbb{Z})$, deci $detX\ \epsilon\ \mathbb{Z}$; pe de alta parte, din $(\det X)^3 = -2$, rezulta $detX = \sqrt[3]{-2} \notin \mathbb{Z}$.

(iii) (C-H): $A^2 - (1+4)A + (4-6)I_2 = 0_2$, adica
$A^2 - 5A - 2I_2 = 0_2 \Leftrightarrow A^2 = 5A + 2I_2$ **(1)**.

Mai departe, utilizand relatia **(1)**, obtinem: $A^3 = A^2 A = 5A^2 + 2A = 5(5A + 2I_2) + 2A = 27A + 10\ I_2$, deci $(a_3, b_3) = (27,10)$.

(iv) Din datele problemei, primim:

$A^n = a_n A + b_n I_2$ **(2)**

$A^{n+1} = a_{n+1} A + b_{n+1} I_2$ **(3)**

Mai departe, utilizand relatiile **(2)** si **(1)**, obtinem:

$A^{n+1} = A^n A = a_n A^2 + b_n A = a_n(5A + 2I_2) + b_n A$, deci

$A^{n+1} = (5a_n + b_n)A + 2a_n I_2$ **(4)**.

Din **(3)** si **(4)** rezulta $a_{n+1} = 5a_n + b_n$ si $b_{n+1} = 2b_n$.

Problema 1.2 (Concurs Licee partenere, 2016)

Fie matricea $A = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R})$. Pentru fiecare pereche (a,b) de numere reale definim matricea $X(a,b) = aA - bI_2 \ \epsilon\ \mathcal{M}_2(\mathbb{R})$.

(i) $(X(a,b))^2 = 0_2$ daca si numai daca

A: $a = b = 0$; ***B***: $b = 3a$; **C**: $b = -3a$; ***D***: $b^2 = 9a^2$; **E**: alt raspuns

(ii) Perechea (a,b) de numere reale pentru care $X(a,b) = A^2$ este:

A: (6,9); ***B***: $(6,-9)$; **C**: $(-6,-9)$; ***D***: $(-9,6)$; **E**: $(9,-6)$

(iii) Perechea (a,b) de numere reale pentru care $X(a,b) = A^{-1}$ este:

A: $\left(\frac{1}{6}.\frac{1}{9}\right)$; ***B***: $\left(-\frac{1}{9}.\frac{1}{6}\right)$; **C**: $\left(\frac{1}{6}.-\frac{2}{9}\right)$; ***D***: $\left(\frac{1}{6}.-\frac{2}{3}\right)$; **E**: $\left(-\frac{1}{9}.-\frac{2}{3}\right)$

(iv) Daca sirurile (a_n) si (b_n) verifica relatia $X(a_n, b_n) = A^n$ pentru orice numar natural n, atunci valoarea limitei $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ este:

A: $\frac{1}{6}$; ***B***: $\frac{1}{2}$; **C**: $\frac{2}{3}$; ***D***: $\frac{1}{3}$; **E**: $\frac{3}{2}$

Rezolvare

(i) Ecuatia Cayley-Hamilton este

$$A^2 - 6A + 9I_2 = 0_2 \Leftrightarrow A^2 = 6A - 9I_2 \ (1).$$

Pe de alta parte, $(X(a,b))^2 = (aA - bI_2)^2 = a^2A^2 - 2abA + b^2I_2$ (2).

Din (1) si (2) rezulta $(X(a,b))^2 = (6a^2 - 2ab)A - (9a^2 - b^2)I_2$.

Astfel, $(X(a,b))^2 = 0_2$ revine la egalitatile $6a^2 - 2ab = 0$ *si* $9a^2 - b^2=0$, i.e. $2a(3a - b) = 0$ si $(3a - b)(3a + b) = 0$, adica $b = 3a$.

(ii) Din relatia (1), deducem (a,b) = (6,9).

(iii) Egalitatea data revine la $aA - bI_2 = A^{-1}$. Prin inmultire cu matricea A obtinem $aA^2 - bA = I_2$, de unde primim, via (2)

$$(6a - b)A - (9a + 1)I_2 = 0_2, \text{ deci } (a, b) = \left(-\frac{1}{9} . -\frac{2}{3}\right).$$

(iv) Din relatia data deducem $A^n = a_n A - b_n I_2$ (3).

Scriem relatia (1) sub forma $(A - 3I_2)^2 = 0_2$. (4)

Pe de alta parte, cu notatia $B = A - 3I_2$, din (4) rezulta

$B^n = 0_2$, pentru orice $n \geq 2$ (5)

Mai departe, avem succesiv, utilizand (5):

$$A^n = (3I_2 + B)^n = C_n^0 (3I_2)^n + C_n^1 (3I_2)^{n-1} B + C_n^2 (3I_2)^{n-2} B^2 + + \cdots + C_n^n B^n = 3^n I_2 + n3^{n-1} B = 3^n I_2 + n3^{n-1} (A - 3I_2), \quad \text{i.e.}$$

$$A^n = n3^{n-1} A - 3^n (n-1) I_2 \quad (6).$$

Din (3) si (6) primim $a_n = n3^{n-1}$ si $b_n = 3^n(n-1)$, asadar

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3}$$

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2. *PUTEREA n A UNEI MATRICE PATRATICE*

2.1. Matrice de rotatie

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R}) \quad \overset{ind}{\Longrightarrow}$$

$$A^n = \begin{pmatrix} \cos nt & -\sin nt \\ \sin nt & \cos nt \end{pmatrix}, n \epsilon \mathbb{Z}$$

Generalizare

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R}) \ ; \ r = \sqrt{a^2 + b^2}$$

$$A = r\begin{pmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{pmatrix}$$

Deoarece $\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1$, exista $t \epsilon [0,2\pi)$ astfel incat

$$\frac{a}{r} = \cos t \;\; si \;\; \frac{b}{r} = \sin t,$$ deci

$A = r\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, asadar $A^n = r^n\begin{pmatrix} \cos nt & -\sin nt \\ \sin nt & \cos nt \end{pmatrix}$.

Problema 2.1 (Admitere 2018 ; Problemele 705,706,707/TG2025)

Fie $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R})$ si $A^n = \begin{pmatrix} x_n & -y_n \\ y_n & x_n \end{pmatrix}$, $n \epsilon \mathbb{N}$.

(**i**) $2A - A^2$ este:

A: $A+I_2$; ***B***: I_2; **C**: $2I_2$; ***D***: 0_2; **E**: $A - I_2$

(**ii**) A^{48} este:

A: 0_2; ***B***: $2^{12}I_2$; **C**: $2^{48}I_2$; ; ***D***: $2^{48}A$; **E**: $2^{24}I_2$

(iii) $\frac{x_{10}^2+y_{10}^2}{x_8^2+y_8^2}$ este:

A: 16; ***B***: 2; **C**: 8; ***D***: 4; **E**: 1

Rezolvare

(i) Din ecuatia (C-H) obtinem:

$$A^2 - 2A + 2I_2 = 0_2 \Leftrightarrow 2A - A^2 = 2I_2 \;.$$

(ii) ***Metoda 1.*** Scriem A sub forma unei matrice de rotatie generalizate:

$A = \sqrt{1+1}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \sqrt{2}\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$, de unde obtinem:

$$A^n = (\sqrt{2})^n \begin{pmatrix} \cos\frac{n\pi}{4} & -\sin\frac{n\pi}{4} \\ \sin\frac{n\pi}{4} & \cos\frac{n\pi}{4} \end{pmatrix}$$

Astfel, $A^{48} = 2^{24}\begin{pmatrix} \cos 12\pi & -\sin 12\pi \\ \sin 12\pi & \cos 16\pi \end{pmatrix} = 2^{24}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$ $2^{24}\, I_2$.

Metoda 2. Calculand succesiv puterile lui A, obtinem $A^4 = -4I_2$, deci $A^{48} = (A^4)^{12} = 2^{24}\, I_2$.

(iii) ***Metoda 1.*** Din datele problemei si **(ii)**, **Metoda 1**, primim:

$x_n = (\sqrt{2})^n \cos\frac{n\pi}{4}$ si $y_n = (\sqrt{2})^n \sin\frac{n\pi}{4}$, asadar

$x_n^2 + y_n^2 = 2^n$. Astfel, $\frac{x_{10}^2+y_{10}^2}{x_8^2+x_8^2} = \frac{2^{10}}{2^8} = 2^2 = 4$.

Metoda 2. Observam ca $\det(A^n) = a_n^2 + b_n^2$, deci $\frac{x_{10}^2+y_{10}^2}{x_8^2+x_8^2} = \frac{\det(A^{10})}{\det(A^8)} = \frac{(\det A)^{10}}{(\det A)^8} = (\det A)^2 = 2^2 = 4$.

2.2 Utilizarea ecuatiei Cayley-Hamilton (C-H)

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{C})$

$\xRightarrow{calcul} \begin{cases} A^2 - (a+d)A + (ad-bc)I_2 = 0_2 \\ \quad A^2 - (TrA)A + \det(A)I_2 = 0_2 \end{cases}$

2.2.1. Daca ***det A = 0***, atunci $A^2 = (TrA)A = (a+d)A$ $\xRightarrow{ind}$ $A^n = (a+d)^{n-1}A = (TrA)^{n-1}A\,,\ n \geq 2.$

2.2.2. Daca ***TrA = a+d = 0***, atunci $A^2 = (bc-ad)I_2 \xRightarrow{ind}$

$$A^{2n} = (bc-ad)^n I_2 \qquad si$$

$$A^{2n+1} = (bc-ad)^n A$$

2.3. Utilizarea unor formule de calcul matriceal

Fie A, B $\boldsymbol{\epsilon \mathcal{M}_k}(\mathbb{C})$***. Daca AB=BA, atunci:***

$(\boldsymbol{i})$ $A^n - B^n$
$= (A-B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + B^{n-1})$

(ii) $A^{2n+1} + B^{2n+1} =$

$(A+B)(A^{2n} - A^{2n-1}B + A^{2n-2}B^2 - A^{2n-3}B^3 + \cdots + B^{2n})$

$(\boldsymbol{iii})$ $\boldsymbol{Binomul\ lui\ Newton:}$ $(A+B)^n = \sum_{k=0}^{n} C_n^k A^{n-k}B^k$,

unde $n \geq 1$ si $A^0 \overset{\text{def}}{=} I_k$.

OBS. Egalitatile de mai sus au loc daca, de exemplu, $B = \alpha I_k,\ \alpha \epsilon \mathbb{C}$

2.3.1. Daca $A^2 + \alpha A + \alpha^2 I_k = 0_k$, atunci $A^3 = \alpha^3 I_k$, deci

$$\begin{cases} A^{3n} = \alpha^{3n} I_k \\ A^{3n+1} = \alpha^{3n} A \\ A^{3n+2} = \alpha^{3n} A^2 \end{cases}$$

Intr-adevar,

$$A^3 - (\alpha I_k)^3 = (A - \alpha I_k)(A^2 + \alpha A + \alpha^2 I_k) = 0_k$$

2.3.2. Daca $A^2 - \alpha A + \alpha^2 I_k = 0_{k,}$ atunci $A^3 = -\alpha^3 I_k$, deci

$$\begin{cases} A^{3n} = (-1)^n \alpha^{3n} I_k \\ A^{3n+1} = (-1)^n \alpha^{3n} A \\ A^{3n+2} = (-1)^n \alpha^{3n} A^2 \end{cases}$$

Similar,

$$A^3 + (\alpha I_k)^3 = (A + \alpha I_k)(A^2 - \alpha A + \alpha^2 I_k) = 0_k$$

2.3.3. Daca $A^n + \alpha A^{n-1} + \alpha^2 A^{n-2} + \cdots + \alpha^n I_k = 0_k$, atunci $A^{n+1} = \alpha^{n+1} I_k$

2.3.4 Daca $A^{2n} - \alpha A^{2n-1} + \alpha^2 A^{2n-2} - \alpha^3 A^{2n-3} + \cdots + \alpha^{2n} I_k = 0_k$, atunci $A^{2n+1} = -\alpha^{2n+1} I_k$

Problema 2.2 (Simulare 2023; Problemele 908, 909, 910/TG 2025)

Fie $a \in \mathbb{C}$ si fie matricea $A = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$.

(i) Multimea valorilor lui a pentru care $A^2 - 2A + I_2 = 0_2$ este:

A: $\{1, -1\}$; ***B***: $\{i, -i\}$; **C**: $\{0\}$; ***D***: $\{\sqrt{2}, -\sqrt{2}\}$; **E**: Φ

(ii) Daca $a{=}i$, atunci numarul solutiilor $X \epsilon \mathcal{M}_2(\mathbb{C})$ ale ecuatiei $AX = I_2$ este:

A: 0 ; ***B***: 1 ; **C**: 2 ; ***D***: 4; **E**: ∞

(iii) Daca $a{=}i$, atunci A^{100} este:

A: $2^{99}A$; ***B***: $2^{100}A$; **C**: $-2^{99}A$; ; ***D***: - $2^{100}A$; **E**: O_2

Rezolvare

(i) Ecuatia Cayley-Hamilton este

$A^2 - 2A + (1 + a^2) = O_2$, de unde rezulta $1 + a^2 = 1$, deci $a = 0.$

(ii) ***Metoda 1.*** Fie $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. Ecuatia $AX = I_2$ conduce la sistemul de ecuatii

$$\begin{cases} x + iz = 1 \\ y + it = 0 \\ -ix + z = 0 \\ -iy + t = 1 \end{cases}$$

Din prima ecuatie rezulta $x = 1 - iz$, care inlocuita in ecuatia a treia furnizeaza contradictia $-i - z + z = 0$. Asadar, ecuatia matriceala $AX = I_2$ ***nu*** are solutii.

Metoda 2 Din egalitatea $AX = I_2$ deducem $\det(AX) = \det(I_2)$, i.e. $(\det A)(detX) = 1$, ceea ce conduce la contradictia $0.(detX) = 1$, asadar ecuatia matriceala $AX = I_2$ ***nu*** are solutii.

(iii) Daca $a{=}i$, atunci $\det A = 0$, de unde (conform observatiei ***2.2.1.*** de mai sus) obtinem $A^n = (a + d)^{n-1}A = (TrA)^{n-1}A$, deci $A^{100} = 2^{99}A$

Problema 2.3 (Simulare 2021 ; Problemele 788, 789, 790/TG 2025)

Se considera matricea $\boldsymbol{A} = \begin{pmatrix} 3 & -4 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \epsilon \mathcal{M}_3(\mathbb{R})$.

(i) Determinantul matricei A este

A: 1 ; ***B***: 0 ; **C**: 2 ; ***D***: 7; **E**: 3

(ii) $(A - I_3)^2$ este:

A: I_3 ; ***B***: 0_3 ; **C**: A ; ***D***: $A - I_3$; **E**: $-I_3$

(iii) A^{2021} este:

A: $\boldsymbol{A} - I_3$; ***B***: 0_3 ; **C**: $A + 2020\, I_3$; ***D***: $2021A - 2020I_3$; **E**: $2020A + 2021I_3$

Rezolvare

(i) Calcul direct

(ii) $A - I_3 = \begin{pmatrix} 2 & -4 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, urmat de calcul direct.

(iii) ***Metoda1.***

Din **(ii)** rezulta $A^2 = 2A - I_3$, de unde deducem succesiv

$$A^3 = A^2.A = (2A - I_3)A = 2(2A - I_3) - A = 3A - 2I_3$$

$$A^4 = A^3.A = (3A - 2I_3)A = 3(2A - I_3) - 2A = 4A - 3I_3$$

Prin inductie, $A^n = nA - (n-1)I_3$, deci $A^{2021} = 2021\,A - 2020\,I_3$.

Metoda 2. Notand $B = A - I_3$, din **(ii)** rezulta

$B^n = 0_3$, pentru orice $n \geq 2$ (1).

Mai departe, utilizand (1) obtinem:

$$A^n = (I_3 + B)^n = C_n^0 (I_3)^n + C_n^1 (I_3)^{n-1} B + C_n^2 (I_3)^{n-2} B^2 + \\ + \cdots + C_n^n B^n = I_3 + n\,B = I_3 + n\,(A - I_3), \quad \text{i.e.} \\ A^n = n\,A - (n-1)I_3 .$$

Problema 2.4 (Admitere 2019; Problema 756/TG 2025)

Fie $A \in \mathcal{M}_2(\mathbb{C})$ o matrice inversabila astfel incat $A + A^{-1} = I_2$.

Matricea $I_2 + A + A^2 + A^3 + \ldots + A^{2019}$ este:

A: $\mathbf{2}A - I_2$; ***B***: $\mathbf{2}A + I_2$; **C**: $-\mathbf{2}A + I_2$; ***D***: $-\mathbf{2}A - I_2$; **E**: $A + I_2$

Rezolvare.

Inmultind egalitatea data cu A, obtinem: $A^2 + I_2 = A$, deci

$$A^2 = A - I_2 \quad \mathbf{(1)}$$

De aici, obtinem :

$$A^3 = A^2.A \overset{(\mathbf{1})}{\Longrightarrow} A^3 = (A - I_2)A = A^2 - A \overset{(\mathbf{1})}{\Longrightarrow} A^3 = -I_2$$

Obs. Egalitatea $A^3 = -I_2$ se putea deduce, similar cu ***Ex. 3.5***, astfel: $A^3 + I_2 = A^3 + (I_2)^3 = (A + I_2)(A^2 - A + I_2) = (A + I_2).0_2 = 0_2$, utilizand **(1)**.

Asadar,

$$A^3 = -I_2 \Rightarrow A^4 = A^3.A = -A,\ A^5 = -A^2,\ A^6 = I_2 \quad \mathbf{(2)}$$

Astfel, din (2) rezulta:

$$I_2 + A + A^2 + A^3 + A^4 + A^5 = I_2 + A + A^2 - I_2 - A - A^2 = 0_2 \quad \mathbf{(3)}$$

Mai departe obtinem, succesiv:

$$I_2 + A + A^2 + A^3 + \ldots + A^{2019} = (I_2 + A + A^2 + A^3 + A^4 + A^5) + (A^6 + A^7 + A^8 + A^9 + A^{10} + A^{11}) + \cdots + (A^{2010} + A^{2011} + A^{2012} + A^{2013} + A^{2014} + A^{2015}) + (A^{2016} + A^{2017} + A^{2018} + A^{2019}) = (I_2 + A + A^2 + A^3 + A^4 + A^5) + A^6(I_2 + A + A^2 + A^3 + A^4 + A^5) + \cdots + A^{2010}(I_2 + A + A^2 + A^3 + A^4 + A^5) + A^{2016}(I_2 + A + A^2 + A^3)$$

$\xRightarrow{(3)\&(2)} I_2 + A + A^2 + A^3 + \ldots + A^{2019} = 0_2 + A^6.0_2 + \cdots + A^{2010}.0_2 + (A^6)^{372}(I_2 + A + A^2 + A^3) = I_2(I_2 + A + A^2 + A^3) = I_2 + A + A^2 + A^3$

$\xRightarrow{(1)\&(2)} I_2 + A + A^2 + A^3 + \ldots + A^{2019} = I_2 + A + A^2 + A^3 = I_2 + A + A - I_2 - I_2 = 2A - I_2.$

Problema 2.5. (Simulare 2019; Problemele 730, 731, 732/TG 2025)

Fie ε radacina pozitiva a ecuatiei $x^2 - x - 1 = 0$ si

$$A = \begin{pmatrix} 1-\varepsilon & \frac{1}{\varepsilon} \\ 1 & \varepsilon - 1 \end{pmatrix} \epsilon\, \mathcal{M}_2(\mathbb{R}).$$

(i) ε^3 este:

A: $\varepsilon - 2$; ***B***: **2**$\varepsilon - 1$; **C**: **2**$\varepsilon + 1$; ***D***: $-\varepsilon + 2$: 2; **E**: ε

(ii) det(A^{2019}) este:

A: 1; ***B***: 0; **C**: 2019; ***D***: -1; **E**: 2

(iii) Matricea A^{2019} este:

A: εI_2; ***B***: $-$ A; **C**: I_2; ***D***: $-\varepsilon I_2$; **E**: A

Rezolvare.

(i) Din $\varepsilon^2 - \varepsilon - 1 = 0$ rezulta

$$\varepsilon^2 = \varepsilon + 1 \quad (\mathbf{1}),$$

de unde primim $\varepsilon^3 = \varepsilon^2.\varepsilon = (\varepsilon + 1).\varepsilon = \varepsilon^2 + \varepsilon = \varepsilon + 1 + \varepsilon = 2\varepsilon + 1.$

(ii) Din **(1)** **,** prin inmultire cu $\frac{1}{\varepsilon}$, primim $\varepsilon = 1 + \frac{1}{\varepsilon}$**,** i.e.

$$\varepsilon - 1 = \frac{1}{\varepsilon} \qquad \mathbf{(2)}$$

Astfel, $\boldsymbol{detA} = -(\varepsilon-1)^2 - \frac{1}{\varepsilon} \xRightarrow{(\mathbf{2})\&(\mathbf{1})} \boldsymbol{detA} = -\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} = -\frac{\varepsilon+1}{\varepsilon^2} = -1.$

(iii) Ecuatia ***(C-H)*** asociata matricei A este:

$A^2 - 0.A + (-1)I_2 = 0_2$, i.e. $A^2 = I_2$.

Astfel, $A^{2019} = A^{2018}.A = (A^2)^{1009}.A = (I_2)^{1009}.A = I_2.A = A$.

3. *SISTEME DE ECUATII LINIARE*

Se considera un sistem liniar de ***m*** ecuatii cu ***n*** necunoscute. Fie $\boldsymbol{A} \in \mathcal{M}_{m,n}(\mathbb{C})$ matricea sistemului si $\overline{\boldsymbol{A}} \in \mathcal{M}_{m,n+1}(\mathbb{C})$ matricea sa extinsa.

Notiuni utilizate

Sistem compatibil $\overset{\boldsymbol{def}}{\Longleftrightarrow}$ ***are cel putin o solutie***

Sistem compatibil determinat $\overset{\boldsymbol{def}}{\Longleftrightarrow}$ ***are exact o solutie*** $\Longleftrightarrow$ ***are solutie unica***

Sistem incompatibil $\overset{\boldsymbol{def}}{\Longleftrightarrow}$ ***nu are nici o solutie***

Sistem compatibil nedeterminat $\overset{\boldsymbol{def}}{\Longleftrightarrow}$ ***are o infinitate de solutii***

OBS. Daca sistemul liniar are coeficientii intr-un corp comutativ ***K,*** de exemplu $\boldsymbol{K} = \mathbb{Z}_p$, atunci sistemul este ***compatibil nedeterminat*** daca ***are cel putin doua solutii.***

3.1. Sisteme liniare de n ecuatii cu n necunoscute

In acest caz, $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{C})$.

(i) Sistemul este ***compatibil determinat*** $\Longleftrightarrow$

$\Longleftrightarrow$ ***det(A)≠ 0*** $\Longleftrightarrow$ ***rang(A) = rang($\bar{A}$) = n***

(ii) Sistemul este ***compatibil nedeterminat*** $\Longleftrightarrow$

$\Longleftrightarrow$ ***rang(A)= rang($\bar{A}$) < n*** $\Longleftrightarrow$

$\Longleftrightarrow$***det(A)= 0 si rang(A)= rang($\bar{A}$)***

(iii) Sistemul este ***incompatibil*** $\Longleftrightarrow$ ***rang(A) < rang($\bar{A}$)***

Obs. ***rang(A) < rang($\bar{A}$)*** $\Longrightarrow$ ***det(A)= 0***

3.2. Sisteme liniare si omogene de n ecuatii cu n necunoscute

Fie $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{C})$ matricea sistemului **.**

(i) Sistemul este ***compatibil determinat (adica are numai solutia nula sau banala)*** $\Longleftrightarrow$ ***det(A)≠ 0***

(ii) Sistemul este ***compatibil nedeterminat (are si solutii nenule)*** $\Longleftrightarrow$ ***det(A)= 0.***

3.3. Sisteme liniare de m ecuatii cu n necunoscute

(i) Sistemul este ***compatibil*** $\Longleftrightarrow$ ***rang(A) = rang(*** $\bar{A}$ ***)***
(Kronecker-Capelli)

(ii) Sistemul este ***compatibil determinat*** $\Longleftrightarrow$
rang(A) = rang($\bar{A}$ ***) = n***
Obs. ***rang(A) = rang(*** $\bar{A}$ ***)= n*** $\Longrightarrow$ ***m≥ n***

(iii) Sistemul este ***compatibil nedeterminat*** $\Longleftrightarrow$
rang(A)= rang($\bar{A}$ ***) < n***

(iv) Sistemul este ***incompatibil*** $\Longleftrightarrow$ ***rang(A) < rang(*** $\bar{A}$ ***)***

3.4. Sisteme liniare si omogene de m ecuatii cu n necunoscute

(i) Sistemul este ***compatibil determinat (adica are numai solutia nula sau banala)*** $\Longleftrightarrow$ ***rang(A)=n***

(ii) ***(ii)*** Sistemul este ***compatibil nedeterminat (are si solutii nenule)*** $\Longleftrightarrow$ ***rang(A) < n***

Problema 3.1 (Admitere 2023; Problemele 940, 941, 942/TG 2025)

Pentru $a \in \mathbb{R}$ se considera sistemul de ecuatii

$$\begin{cases} x + ay + z = 1 \\ -2x + az = -2 \\ -x + 3y - z = -1 \end{cases},$$

in necunoscutele *x, y, z*, iar prin *A* se noteaza matricea sistemului.

(i) Multimea valorilor lui *a* pentru care *rang A = 3* este:

A: $\mathbb{R}\backslash\{-3,-2\}$; ***B***: Φ; **C**: $\{-3,-2\}$; ***D***: $\{2,3\}$; **E**: $\mathbb{R}\backslash\{2,3\}$

(ii) Numarul valorilor lui a pentru care sistemul admite o infinitate de solutii este:

A: 0; ***B***: 1; **C**: 2 ; ***D***: 3 ; **E**: ∞

(iii) Numarul valorilor lui a pentru care sistemul admite solutie, cu x, y, z in progresie aritmetica, in aceasta ordine, este:

A: 0; ***B***: 1; **C**: 2 ; ***D***: 4 ; **E**: ∞

Rezolvare.

(i) Fie $\boldsymbol{A}=\begin{pmatrix}1 & a & 1\\ -2 & 0 & a\\ -1 & 3 & -1\end{pmatrix}$ matricea sistemului dat.

Avem $rang\ A=3 \Leftrightarrow det(A)\neq 0$.

Adunand linia 1 la linia 3 , primim: $\det(A)=\begin{vmatrix}1 & a & -1\\ -2 & 0 & -a\\ 0 & a+3 & -0\end{vmatrix}=$ $-(a+3)(a+2)$, deci $a\in\mathbb{R}\backslash\{-3,-2\}$.

(ii) Sistemul trebuie sa fie compatibil nedeterminat, deci

$$rang\ A=rang\bar{A}<\mathbf{3}.$$

Din $rang\ A<3$ rezulta det(A) = 0, asadar $a\in\{-3,-2\}$.

Cazul 1: $a=-3$.

Observam ca $rang\ A=2$, determinantul principal fiind, de exemplu

$\det\begin{pmatrix}-2 & 0\\ -1 & 3\end{pmatrix}.=-6\neq 0$. Matricea extinsa a sistemului este

$$\bar{\boldsymbol{A}}=\begin{pmatrix}1 & -3 & 1 & 1\\ -2 & 0 & -3 & -2\\ -1 & 3 & -1 & -1\end{pmatrix}.$$

Conditia $rang\,\overline{A} = 2$ revine la $\Delta_c = \begin{vmatrix} 1 & -3 & 1 \\ -2 & 0 & -2 \\ -1 & 3 & -1 \end{vmatrix} = 0$, ceea ce este adevarat, deoarece linia 3 este opusa liniei 1.

Asadar, pentru $a = -3$, sistemul admite o infinitate de solutii.

Cazul 2: $a = -2$.

Observam ca $rang\ A = 2$, determinantul principal fiind, de exemplu $\det\begin{pmatrix} -2 & 0 \\ -1 & 3 \end{pmatrix}. = -6 \neq 0$. Matricea extinsa a sistemului este

$$\overline{\boldsymbol{A}} = \begin{pmatrix} 1 & -2 & 1 & 1 \\ -2 & 0 & -2 & -2 \\ -1 & 3 & -1 & -1 \end{pmatrix}.$$

Conditia $rang\,\overline{A} = 2$ revine la $\Delta_c = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 0 & -2 \\ -1 & 3 & -1 \end{vmatrix} = 0$, ceea ce este adevarat, deoarece coloana 3 este identica cu coloana 1.

Asadar, pentru $a = -2$, sistemul admite, de asemenea, o infinitate de solutii.

In concluzie, numarul valorilor lui a pentru care sistemul admite o infinitate de solutii este 2 $(a = -2\ \ si\ \ a = -3)$.

Observatie. Pentru $a \in \mathbb{R}\backslash\{-3, -2\}$ ssitemul este compatibil determinat (are solutie unica), iar pentru $a \in \{-3, -2\}$ sistemul este compatibil nedeterminat . Nu exista valori ale lui ***a*** pentru care sistemul sa fie incompatibil.

(iii) Conditia de progresie aritmetica este

$$y = \frac{x+z}{2} \Leftrightarrow \boldsymbol{x} - 2y + z = 0 \qquad \textit{(1)}$$

Metoda 1.

Cazul 1. Daca $\boldsymbol{a} \in \mathbb{R} \backslash \{\boldsymbol{-3}, \boldsymbol{-2}\}$, atunci sistemul are solutie unica, care se poate determina prin regula lui Cramer sau utilizand combinatii liniare de ecuatii (de exemplu, adunand prima si a treia ecuatie).

Mai direct, observam ca , in cazul general, coloana necunoscutei ***x*** coincide cu coloana termenilor liberi, ceea ce arata ca o solutie a sistemului (in cazul nostru ***unica solutie***) este ***x = 1, y = 0, z = 0.*** Deoarece conditia $y = \frac{x+z}{2}$ nu este indeplinita, deducem ca in acest caz, nu exista $\boldsymbol{a}$ pentru care sa avem progresie artmetica.

Cazul 2. Fie $\boldsymbol{a = -3.}$

Rezolvam sistemul (compatibil nedeterminat)

$$\begin{cases} x - 3y + z = 1 \\ -2\ x - \quad 3z\ = -2 \\ -x + 3y - z = -1 \end{cases}$$

Luam primele doua ecuatii drept ecuatii principale si pe ***z*** drept necunoscuta secundara.

Obtinem $x = 1 - \frac{3t}{2}$, $y = -\frac{t}{6}$, $z = t$, iar din conditia de progresie aritmetica 2y = x + z , rezulta $t = 6$, deci $x = -8$, $y = -1, z = 6$.

Asadar $\boldsymbol{a = -3}$ este solutie a problemei.

Cazul 3. Fie $\boldsymbol{a = -2.}$

Rezolvam sistemul (compatibil nedeterminat)

$$\begin{cases} x - 2y + z = 1 \\ -2\ x - \quad 2z\ = -2 \\ -x + 3y - z = -1 \end{cases}$$

Adunand prima si ultima ecuatie, primim $y = 0$, iar sistemul se reduce la ecuatia $x + z = 1$. Intrucat conditia de progresie aritmetica 2y = x + z i.e. 0 = 1 nu este indeplinita, rezulta ca $a = -2$ ***nu*** este solutie a problemei.

In concluzie, ***singura*** valoare acceptabila este $\boldsymbol{a = -3}$**.**

Raspuns corect B.

Metoda 2.

Atasam sistemului dat conditia (1) de progresie aritmetica.

Obtinem sistemul de 4 ecuatii cu 3 necunoscute x, y, z si parametrul a:

$$\textbf{(S)}\begin{cases} x + ay + z = 1 \\ -2x + \quad az = -2 \\ -\ x + 3y - z = -1 \\ x - 2y + z = 0 \end{cases}$$

Fie $\boldsymbol{\mathcal{A}} = \begin{pmatrix} 1 & a & 1 \\ -2 & 0 & a \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$ matricea sistemului si

$\overline{\boldsymbol{\mathcal{A}}} = \begin{pmatrix} 1 & a & 1 & 1 \\ -2 & 0 & a & -2 \\ -1 & 3 & -1 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix}$ matricea sa extinsa.

Observam ca $\boldsymbol{\mathcal{A}} \in \mathcal{M}_{4,3}(\mathbb{R})$ **,** deci

$$\boldsymbol{rang(\mathcal{A}) \le 3} \quad (\mathbf{2})\ .$$

Pe de alta parte, conditia de compatibilitate a sistemului (impusa de cerintele problemei), anume

$$\boldsymbol{rang(\mathcal{A}) = rang(\overline{\mathcal{A}})} \quad (\mathbf{3})\ ,$$

impreuna cu relatia $(\mathbf{2})$, conduce la egalitatea

$$\boldsymbol{det(\overline{\mathcal{A}}) = 0} \quad \textbf{(4)}$$

Intr-adevar, daca $\det(\bar{\mathcal{A}}) \neq 0$, atunci $\operatorname{rang}(\bar{\mathcal{A}}) = 4$, ceea ce, alaturi de relatiile $(\mathbf{2})$ si $(\mathbf{3})$, furnizeaza contradictia $\boldsymbol{rang(\mathcal{A})} = \mathbf{4} \leq \mathbf{3}$.

Sa calculam $\det(\bar{\mathcal{A}})$. Adunand linia 3 la liniile 1 si 4, obtinem

$$\boldsymbol{det(\bar{\mathcal{A}})} = \begin{vmatrix} 0 & a+3 & 0 & 0 \\ -2 & 0 & a & -2 \\ -1 & 3 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{vmatrix} =$$

$$(-1)^{1+2}.(a+3).\begin{vmatrix} -2 & a & -2 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = (a+3)(a+2) \quad (\mathbf{5})$$

Din (4) si (5), rezulta $a \in \{-3, -2\}$.

Verificam conditia $rang(\mathcal{A}) = rang(\bar{\mathcal{A}})$

Pentru $\boldsymbol{a = -3}$, $rang(\mathcal{A}) = 3$, alegand $\Delta_p = \begin{vmatrix} 1 & -3 & 1 \\ -2 & 0 & -3 \\ 1 & -2 & 1 \end{vmatrix} = 1 \neq 0.$

Pe de alta parte, $\det(\bar{\mathcal{A}}) = 0$, deci $rang(\mathcal{A}) = rang(\bar{\mathcal{A}}) = 3$.

Pentru $\boldsymbol{a = -2}$, sistemul corespunzator **(S)** de mai sus este:

$$\begin{cases} x - 2y + z = 1 \\ -2x \quad - \quad 2z = -2 \\ -\ x + 3y - z = -1 \\ x - 2y + z = 0\ , \end{cases}$$

iar prima si ultima sa ecuatie conduc la contradictia $0 = 1$.

De altfel, se poate constata ca $rang(\mathcal{A}) = 2$, iar $rang(\bar{\mathcal{A}}) = 3$,

deci sistemul este incompatibil.

In concluzie, ***singura*** valoare a lui a pentru care sistemul dat admite solutie, cu x, y, z in progresie aritmetica, in aceasta ordine, este $\boldsymbol{a = -3}$.

Raspuns corect B.

4. DETERMINANTI VANDERMONDE

$$V(a,b,c) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

$$V(a,b,c,d) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

$$V(a_1, a_2, \ldots, a_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} =$$

$$= \prod_{1 \le i < j \le n} (a_j - a_i)$$

Problema 4.1 (Problema 185/TG2025)

Multimea solutiilor reale ale ecuatiei

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & -1 & 2 \\ 1 & x^3 & -1 & 8 \\ 1 & x^2 & 1 & 4 \end{vmatrix} = 0 \text{ este:}$$

A: {-1,1,2} ; ***B***: $\mathbb{R}\backslash\{-1,1,2\}$; **C**: $\{-1,1-2\}$; ***D***: Φ ; **E**: D {1}

Rezolvare

Schimband intre ele liniile 3 si 4, obtinem:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & -1 & 2 \\ 1 & x^3 & -1 & 8 \\ 1 & x^2 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & -1 & 2 \\ 1 & x^2 & 1 & 4 \\ 1 & x^3 & -1 & 8 \end{vmatrix} = -V(1, x, -1, 2) =$$

$-1(x-1)(-1-1)(2-1)(-1-x)(2-x)(2+1) = 6(x-1)(x+1)(2-x) = 0,\ deci\ x \in \{-1,1,2\}$. Raspuns corect: **A.**

Problema 4.2 (Problema 48/TG2024)

Determinantul $\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{vmatrix}$ este:

A: 16i; ***B***: $-$ 16i ; **C**: 16; ***D***: -16 ; **E**: 0

Rezolvare

$\Delta = V(1, -i, -1, i) = (-i-1)(-1-1)(i-1)(-1+i)(i+i)(i+1) = 16i$; raspuns corect **A.**

5. *PROBLEME PROPUSE*

5.1. (Problema 56/TG2025)

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a+2 & b+2 & c+2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix}, \ a < b < c.$$

A: D= 0; ***B***: D ≤ 0 ; **C**: $D < 0$; ***D***: D > 0 ; **E**: D $= -a^2 - b^2 - c^2$

Rezolvare

Scazand linia 1 din linia 2, obtinem:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a+1 & b+1 & c+1 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix} = V(a+1, b+1, c+1) = (b+1-a-1)(c+1-a-1)(c+1-b-1) = (b-a)(c-a)(c-b) > 0,$$

deci raspuns corect ***D.***

5.2. (Admitere 2006; a se vedea si problemele 177, 178, 179/ TG2025)

Fie $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \epsilon\, \mathcal{M}_2(\mathbb{R})$.

(i) Numarul solutiilor inversabile $X \epsilon\, \mathcal{M}_2(\mathbb{R})$ ale ecuatiei $X^{2006} = A$ este:

A: 2006; ***B***: 1003; **C**: 0; ***D***: ∞; **E**: 2

(ii) Numarul solutiilor $X \epsilon\, \mathcal{M}_2(\mathbb{R})$ ale ecuatiei $X^{2006} = A$ este:

A: 2006; ***B***: 1003; **C**: 0; ***D***: ∞; **E**: 2

(iii) Numarul solutiilor $X \epsilon\, \mathcal{M}_2(\mathbb{C})$ ale ecuatiei $X^{2006} = A$ este:

A: 2006; ***B***: 1003; **C**: 0; ***D***: ∞; **E**: 2

Rezolvare

(i) Trecand la determinant, similar cu **Ex.1.1**, primim ***det(X)=0,*** deci nu exista inversa matricei X.

(ii) Deoarece *det(X)=0*, in conformitate cu ***2.2.1.,*** obtinem

$$X^n = (a+d)^{n-1}X = (TrA)^{n-1}X\ ,\ n \geq 2,$$

unde $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Astfel, ecuatia $X^n = A$ devine

$$(a+d)^{n-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\ ,$$

adica $\begin{cases} a(a+d)^{n-1} = 2 \\ b(a+d)^{n-1} = 3 \\ c(a+d)^{n-1} = 4 \\ d(a+d)^{n-1} = 6 \end{cases}$

Adunand prima si ultima ecuatie, obtinem

$$(a+d)^n = 8 \Leftrightarrow (TrX)^n = TrA\ .$$

Astfel, pentru **(ii),** rezulta $(a+d)^{2006} = 8,$ cu $a, d\, \epsilon \mathbb{R}$**,** asadar $a + d = \pm\sqrt[2006]{8}$, ceea ce conduce la doua solutii (a,b,c,d) ale sistemului de mai sus, deci la doua matrice X.

(iii) Similar cu **(ii)**, ecuatia binoma $(a+d)^{2006} = 8,$ cu $a, d\ \epsilon\ \mathbb{C}$**,** are ***2006*** solutii complexe, ceea ce conduce la ***2006*** solutii (a,b,c,d) ale sistemului de mai sus, deci la ***2006*** matrice X.

5.3. (Admitere 2016; a se vedea si problema 223/ TG2025)

Fie $A \epsilon\ \mathcal{M}_2(\mathbb{R})$ o matrice nenula astfel incat $A^{2016} = 0_2$. Cardinalul multimii $\{A^n / n \geq 1\}$ este:

A: 2016; ***B***: 1008; **C**: 0; ***D***:2; **E**: ∞

Rezolvare

Din $A^{2016} = 0_2$ rezulta $\det(A^{2016}) = \det(0_2)$, deci $(detA)^{2016} = 0$ i.e. $detA = 0$. In conformitate cu ***2.2.1.*** si ipoteza $A^{2016} = 0_2$, obtinem $A^{2016} = (TrA)^{2015}A = 0_2$. Deoarece $A \neq 0_2$, deducem TrA=0.

Din relatiile det A = Tr A = 0, utilizand ecuatia (C-H)

$$A^2 - (TrA)A + \det(A)I_2 = 0_2 ,$$

deducem $A^2 = 0_2$, deci $A^n = 0_2, \forall n \geq 2$.

Astfel, $M = \{A, A^2\} = \{0_2, A\}$, asadar M are 2 elemente.

5.4. (Test admitere ; Problema 672/TG2023)

Numarul solutiilor $X \epsilon \mathcal{M}_2(\mathbb{R})$ ale ecuatiei $X^{25} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ este:

A: 2; ***B***: 0; **C**: 10; ***D***:25; **E**: ∞

Rezolvare

Similar cu ***Ex. 2.2*** primim $\det(X) = det \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$=0 si $(\mathrm{Tr}X)^{25} = Tr \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = 0$, i.e. $TrX = 0$. Din ecuatia (C-H) rezulta X^2=0_2, deci X^{25}=0_2, ceea ce contrazice ipoteza problemei $X^{25} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$.

Asadar, ***nu exista*** nici o matrice $X \epsilon \mathcal{M}_2(\mathbb{R})$ care sa fie solutie a ecuatiei date.

5.5. (Simulare 2018; 679/ TG2025)

Fie $A \epsilon \mathcal{M}_2(\mathbb{R})$ si $\lambda \,\epsilon\, \mathbb{R}$ astfel incat $A^2 - \lambda A + \lambda^2 I_2 = 0_2$.

Matricea A^{2018} este:

A: $\lambda^{2018} I_2$; ***B***: A; **C**: $\lambda^{2016} A^2$; ***D***: $\lambda^2 A^2$; **E**: 0_2

Rezolvare

Avem $A^3 + (\lambda I_2)^3 = (A + \lambda I_2)\,(A^2 - \lambda A + \lambda^2 I_2) =$

$(A + \lambda I_2)\, 0_2 = 0_2$, deci $A^3 = -(\lambda I_2)^3 = -\lambda^3 I_2$.

Mai departe , $A^{2018} = A^{2016} A^2 = (A^3)^{672} A^2 =$
$(-\lambda^3 I_2)^{672} A^2 = (-1)^{672} (\lambda^3)^{672} (I_2)^{672} A^2 = \lambda^{2016} I_2 A^2 =$
$\lambda^{2016} A^2$.

5.6. (Admitere 2011; a se vedea si problema 169/ TG2025)

Fie $A = \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \epsilon\, \mathcal{M}_2(\mathbb{R})$.

(i) Numarul $a \,\epsilon\, \mathbb{R}$ pentru care $aA - A^2 = I_2$ este:

A: 1; ***B***: 3; **C**: 0; ***D***: 2; **E**: -1

(ii) $A + A^{-1}$ este:

A: $2I_2$; ***B***: **3**A ; **C**: A^2; ***D***: 0_2; **E**: Nu exista A^{-1}

(iii) $A^{2011} + (A^{-1})^{2011}$ este:

A: $A + I_2$; ***B***: **3**A ; **C**: A^2; ***D***: 0_2; **E**: **:** $2I_2$

Rezolvare.

(i) Ecuatia (C-H) este: $A^2 - 2A + I_2 = 0_2$, i.e.
$2A - A^2 = I_2$, deci $a = 2$.

(ii) Din $A^2 - 2A + I_2 = 0_2$, prin inmultire cu A^{-1} (inversa exista deoarece *det*A = 1 ≠ 0), rezulta $A^2 A^{-1} - 2AA^{-1} + I_2 A^{-1} = 0_2$, deci $A - 2I_2 + A^{-1} = 0_2$.

Astfel, $A + A^{-1} = 2I_2$.

(iii) ***Metoda 1***

Din $A + A^{-1} = 2I_2$, prin ridicare la patrat, obtinem

$A^2 + (A^{-1})^2 + 2AA^{-1} = 4I_2$, deci $A^2 + (A^{-1})^2 = 2I_2$.

Se demonstreaza prin metoda inductiei matematice relatia

$$A^n + (A^{-1})^n = 2I_2, \forall n \geq 1.$$

Metoda 2

Scriem ecuatia (C-H) de la **(i)** sub forma $(A - I_2)^2 = 0_2$ (1).

Astfel, primim: $(A^k - I_2)^2 = \left(A^k - I_2^k\right)^2 =$

$= [(A - I_2)(A^{k-1} + A^{k-2} + \cdots + A + I_2)]^2 =$

$=(A - I_2)^2(A^{k-1} + A^{k-2} + \cdots + A + I_2)^2$, de unde obtinem, via (2):

$(A^k - I_2)^2 = 0_2$, i.e. $A^{2k} - 2A^k + I_2 = 0_2$.

Prin inmultire cu $(A^{-1})^k$, deducem $A^k + (A^{-1})^k = 2I_2 , \forall k \geq 1$.

5.7. (Problema 21/ TG2025)

Fie P, Q, R functii de grad cel mult 2 si a, b, c numere complexe date.
Se considera determinantii:

$$\Delta_0 = \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(b) & Q(b) & R(b) \\ P(c) & Q(c) & R(c) \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} P(1) & Q(1) & R(1) \\ P(b) & Q(b) & R(b) \\ P(c) & Q(c) & R(c) \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(1) & Q(1) & R(1) \\ P(c) & Q(c) & R(c) \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(b) & Q(b) & R(b) \\ P(1) & Q(1) & R(1) \end{vmatrix}$$

Daca $\Delta_0 = 1$, atunci $\Delta_1 + \Delta_2 + \Delta_3$ este:

A: 0; ***B***: 1; **C**: 3; ***D***: $P(0) + Q(0) + R(0)$; **E**: P(1)Q(1)R(1)

Rezolvare.

Fie $x \in \mathbb{R}$ si

$$H(x) = \begin{vmatrix} P(x) & Q(x) & R(x) \\ P(b) & Q(b) & R(b) \\ P(c) & Q(c) & R(c) \end{vmatrix} + \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(x) & Q(x) & R(x) \\ P(c) & Q(c) & R(c) \end{vmatrix} + \begin{vmatrix} P(a) & Q(a) & R(a) \\ P(b) & Q(b) & R(b) \\ P(x) & Q(x) & R(x) \end{vmatrix}$$

Avem H(a) = $\Delta_0 + 0 + 0 = \Delta_0 = 1$; analog H(b) =1 si H(c) =1 $\mathbf{(1)}$

Pe de alta parte, observam ca H este o functie de gradul doi,

H(x) $= mx^2 + nx + p$ $\mathbf{(2)}$

si $\Delta_1 + \Delta_2 + \Delta_3 = H(1)$ $\mathbf{(3)}$.

Relatiile $\mathbf{(1)}$ & $\mathbf{(2)}$ conduc la sistemul linear $\begin{cases} ma^2 + na + p = 1 \\ mb^2 + nb + p = 1 \\ mc^2 + nc + p = 1 \end{cases}$,

cu necunoscutele m, n, p. Determinantul sistemului este:

$$\Delta = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -V(a,b,c) = -(b-a)(c-a)(c-b)$$

Deoarece $\Delta_0 = 1 \neq 0$, rezulta ca numerele a, b, c sunt distincte intre ele. De aici deducem ca $\Delta \neq 0$, deci sistemul de mai sus are solutie unica, pe care o determinam cu regula lui Cramer.

Astfel, $\Delta_m = 0\ (col\ 1 = col\ 3), \Delta_n = 0\ (col\ 2 = col\ 3)$ si $\Delta_p = \Delta$, deci m=$\frac{\Delta_m}{\Delta} = 0$, n=$\frac{\Delta_n}{\Delta} = 0$ si p = $\frac{\Delta_p}{\Delta}$ = 1, de unde obtinem, via **(2)** ,

H(x) = $mx^2 + nx + p$ = 1 , $\forall x \in \mathbb{R}$.

In final, utilizand **(3)** , obtinem $\Delta_1 + \Delta_2 + \Delta_3 = H(1) = 1.$

5.8 (Admitere 2010 ; a se vedea si problema 63/ TG2025)

Fie $A = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \epsilon \mathcal{M}_2(\mathbb{R})$ si $A^n = \begin{pmatrix} a_n & -b_n \\ b_n & a_n \end{pmatrix}$, $n \epsilon \mathbb{N}$.

(i) A^2 este:

A: 2A; ***B***: 2A + $4I_2$; **C**: I_2; ***D***: 0_2; **E**: 2A − $4I_2$

(ii) A^{48} este:

A: $2^{24}I_2$; ***B***: $-2^{24}I_2$; **C**: I_2; ***D***: $2^{48}I_2$; **E**: $-2^{48}I_2$

(iii) $\frac{a_{20}^2 + b_{20}^2}{a_{10}^2 + b_{10}^2}$ este:

A: 2^{15}; ***B***: 2^5; **C**: 1; ***D***: 2^{10}; **E**: 2^{20}

(iv) Numarul valorilor n ≥ 1 pentru care $A^n = 8I_2$ este:

A: 3; ***B***: 0; **C**: 1; ***D***:2; **E**: ∞

Rezolvare

(i) Din ecuatia (C-H) obtinem:

$A^2 - 2A + 4I_2 = 0_2 \Leftrightarrow A^2 = 2A - 4I_2$.

(ii) ***Metoda 1.*** Scriem A sub forma unei matrice de rotatie generalizate:

$A = \sqrt{1+3}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = 2\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$, de unde obtinem:

$$A^n = 2^n\begin{pmatrix} \cos\frac{n\pi}{3} & -\sin\frac{n\pi}{3} \\ \sin\frac{n\pi}{3} & \cos\frac{n\pi}{3} \end{pmatrix}$$

Astfel, $A^{48} = 2^{48}\begin{pmatrix} \cos 16\pi & -\sin 16\pi \\ \sin 16\pi & \cos 16\pi \end{pmatrix} = 2^{48}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2^{48}\, I_2$.

Metoda 2. Din ecuatia (C-H) rezulta

$A^2 - 2A + 2^2 I_2 = 0_2$, de unde obtinem:

$A^3 + (2I_2)^3 = (A + 2I_2)(A^2 - 2A + 2^2 I_2) = (A + 2I_2)0_2 = 0_2$, deci $A^3 = (-2I_2)^3 = -8I_2$.

Astfel, $A^{48} = (A^3)^{16} = (-2^3 I_2)^{16} = 2^{48}\, I_2$.

(iii) ***Metoda 1.*** Din datele problemei si **(ii)**, **Metoda 1**, primim:

$a_n = 2^n \cos\frac{n\pi}{3}$ si $b_n = 2^n \sin\frac{n\pi}{3}$, asadar

$a_n^2+b_n^2=2^{2n}$. Astfel, $\dfrac{a_{20}^2+b_{20}^2}{a_{18}^2+b_{18}^2}=\dfrac{2^{40}}{2^{20}}=2^{20}$.

Metoda 2. Observam ca $\det(A^n)=a_n^2+b_n^2$, deci

$$\frac{a_{20}^2+b_{20}^2}{a_{18}^2+b_{18}^2}=\frac{\det(A^{20})}{\det(A^{10})}=\frac{(\det A)^{20}}{(\det A)^{10}}=(\det A)^{10}=2^{20}.$$

(iv) ***Metoda 1.***

$$A^n=8I_2 \overset{(ii),\ \ Met.1}{\Longleftrightarrow} 2^n\begin{pmatrix}\cos\frac{n\pi}{3} & -\sin\frac{n\pi}{3}\\ \sin\frac{n\pi}{3} & \cos\frac{n\pi}{3}\end{pmatrix}=8\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$\Leftrightarrow 2^n\cos\frac{n\pi}{3}=8 \wedge 2^n\sin\frac{n\pi}{3}=0 \Leftrightarrow 2^n\cos\frac{n\pi}{3}=8 \ \wedge$ $\frac{n\pi}{3}=k\pi,\ k\epsilon\ \mathbb{N} \Leftrightarrow n=3k,\ 2^{3k}\cos(k\pi)=8 \Leftrightarrow$ $2^{3k}(-1)^k=8 \Leftrightarrow (-8)^k=8,\ k\epsilon\ \mathbb{N}$, ceea ce este imposibil, deoarece $8\notin\{(-8)^k\ /\ k\epsilon\ \mathbb{N}\}$.

Metoda 2. Din (ii), Metoda 2, rezulta $A^3=-8I_2$, de unde deducem

$$\begin{cases} A^{3k}=(-8)^kI_2\\ A^{3k+1}=(-8)^kA\\ A^{3k+2}=(-8)^kA^2\end{cases}$$

In nici una dintre aceste situatii nu putem primi $A^n=8I_2$.

5.9 Admitere 2022 (Problemele 874, , 875, 876 / TG2025)

Fie matricea $A=\begin{pmatrix}2 & 6\\ 1 & 3\end{pmatrix}$.

(i) Valoarea lui $x\epsilon\ \mathbb{R}$ pentru care $A^2=xA$ este:

A: 5 ; ***B***: 1 ; **C**: 2; ***D***: 6; **E**: 3

(ii) A^{2022} este:

A: $5^{2021}A$; ***B***: **5**A; **C**: $5^{2021}I_2$; ***D***: $6^{2021}A$; **E**:

(iii) $\det(A + A^2 + A^3 + \cdots + A^{2022})$ este:

A: 0 ; ***B***: 2022 ; **C**: 5^{4044}; ***D*** $\frac{5^{2023}-1}{4}$: 6; **E**: 6^{2023}

Rezolvare

(i) Din ecuatia (C-H) obtinem

$$A^2 - 5A + 0.I_2 = 0_2 \Leftrightarrow A^2 = 5A.$$

(ii) Din sectiunea 2.2.1, obtinem

$$A^n = (a+d)^{n-1}A = (TrA)^{n-1}A\ ,$$

deci $A^{2022} = 5^{2021}A.$

(iii) Avem $\det(A + A^2 + A^3 + \cdots + A^{2022}) = \det[\mathrm{A}(A + A^2 + A^3 + \cdots + A^{2021})] = \det(\mathrm{A}) . \det(A + A^2 + A^3 + \cdots + A^{2021}) = 0$, deoarece det(A) = 0.

5.10 Admitere 2019 (Problemele 758, 759 / TG2025)

Se considera sistemul de ecuatii liniare

$$\begin{cases} 2x + y + z = 1 \\ x + 2y + z = \mathrm{b} \\ x + y + az = 1 \end{cases}$$

, cu a, b numere reale.

(i) Sistemul este compatibil determinat daca si numai daca :

A: $a \neq \frac{2}{3}$; ***B***: $a = \frac{2}{3}$; **C**: $a \neq \frac{3}{2}$; ***D***: $a = \frac{3}{2}$ 6; **E**: $a \neq 2$

(ii) Sistemul este compatibil nedeterminat daca si numai daca :

A: $a = \frac{2}{3}, b = 2$; ***B***: $a = \frac{2}{3}$, $b \neq 2$; **C**: $a = \frac{3}{2}, b = 2$;

D: $a=\frac{2}{3}$, $b=3$; **E**: $a\neq\frac{2}{3}$, $b=2$

Rezolvare

Fie $\boldsymbol{A}=\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & a\end{pmatrix}$ matricea sistemului dat si

$\bar{\boldsymbol{A}}=\begin{pmatrix}2 & 1 & 1 & 1\\ 1 & 2 & 1 & b\\ 1 & 1 & a & 1\end{pmatrix}$ matricea sa extinsa.

(i) Din **3.1.(i)** avem conditia ***det(A)≠0,*** i.e $3a-2\neq 0$, deci $a\neq\frac{2}{3}$.

(ii) Din **3.1.(ii)** avem conditiile ***det(A)= 0 si rang(A)= rang(*** $\bar{A}$ ***).*** Astfel, $a=\frac{2}{3}$ si $\boldsymbol{rang\ A=2}$, alegand $\Delta_p=\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=3\neq 0$. Mai departe,

rang($\bar{A}$ ***)***= $\boldsymbol{rang\ A=2}\Leftrightarrow\Delta_c=\begin{vmatrix}2 & 1 & 1\\ 1 & 2 & b\\ 1 & 1 & 1\end{vmatrix}=0$, i.e.

$2-b=0\Leftrightarrow b=2$. Asadar, $a=\frac{2}{3}$ si $b=2$.

5.11. (Admitere 2021 ; Problemele 884, 885/ TG2024)

Fie matricea $A=\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}\epsilon\mathcal{M}_2(\mathbb{R})$.

(iv) A^{2022} este:

A: 4^{2021}A ; ***B***: 0_2 ; **C**: $4A$; ***D***: 4^{2022}A ; **E**: $4^{2022}I_2$

(v) Numarul matricelor X $\epsilon\mathcal{M}_2(\mathbb{R})$ care verifica ecuatia $X^{2022}=$ A este:

A: 1 ; ***B***: 2022 ; C: 2 ; ***D***: 0; **E**: 4

Rezolvare

(i) Ecuatia Cayley-Hamilton este

$A^2 = 4A$, de unde rezulta $A^n = 4^{n-1}A$, deci $A^{2022} = 4^{2021}$A.

(ii) Din $X^{2022} = $ A rezulta $\det(X^{2022}) = \det A$, deci $(\det X)^{2022} = 0$.
Astfel, detX = 0 si , in conformitate cu ***3.2.1.,*** obtinem

$$X^n = (a+d)^{n-1}X = (TrA)^{n-1}X\ ,\ n \geq 2,$$

unde $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Ecuatia $X^{2022} = A$ devine

$$(a+d)^{2021}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\ ,\ \text{adica}\quad \begin{cases} a(a+d)^{2021} = 2 \\ b(a+d)^{2021} = -4 \\ c(a+d)^{2021} = -1 \\ d(a+d)^{2021} = 2 \end{cases}$$

Adunand prima si ultima ecuatie, obtinem

$(a+d)^{2022} = 4 \Leftrightarrow (TrX)^{2022} = TrA$, de unde rezulta doua solutii reale,

$a + d = \sqrt[2022]{4}$ si $a + d = -\sqrt[2022]{4}$, deci ecuatia $X^{2022} = A$ are doua solutii in $\mathcal{M}_2(\mathbb{R})$.

Observatie. Ecuatia $(a+d)^{2022} = 4$ are 2022 solutii distincte in $\mathbb{C}$, asadar ecuatia $X^{2022} = A$ are 2022 solutii distincte in $\mathcal{M}_2(\mathbb{C})$.

5.12. (Admitere 2017; Problemele 715, 716, 717/ TG2024)

Se considera sistemul de ecuatii liniare

$$\begin{cases} ax + y + z = -1 \\ x + ay + z = -\text{a} \\ x + y - z = -2 \end{cases}\ ,\ \text{unde}\ a \in \mathbb{R}.$$

(iv) Determinantul sistemului este:

A: a^2; ***B***: $a^2 + 2a - 3$; **C**: $a^2 - 2a + 3$; ***D***: $-a^2 - 2a + 3$; **E**: $2\text{a} + 3$

(v) Sistemul este incompatibil daca si numai daca:

A: $a = -1$; ***B***: $a = 1$; **C**: $alt\ raspuns$; ***D***: $a \in \mathbb{R}\backslash\{-3,1\}$; **E**: $a = -3$

(vi) Numarul valorilor reale ale parametrului $a \in \mathbb{R}$ pentru care sistemul admite solutii (x, y, z), cu x, y, z in progresie aritmetica, in aceasta ordine, este:

A: 0; ***B***: 3; **C**: 1; ***D***: 2; **E**: ∞

Rezolvare.

Fie $\boldsymbol{A} = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & -1 \end{pmatrix}$ matricea sistemului dat si

$\overline{\boldsymbol{A}} = \begin{pmatrix} a & 1 & 1 & -1 \\ 1 & a & 1 & a \\ 1 & 1 & -1 & -2 \end{pmatrix}$ matricea sa extinsa.

(i) Prin calcul direct, rezulta $\boldsymbol{detA} = -a^2 - 2a + 3.$

(ii) Ecuatia detA = 0 are solutiile -3 si 1.

Daca $\boldsymbol{a \in \mathbb{R} \setminus \{-3, 1\}}$, atunci sistemul este compatibil determinat.

Daca $\boldsymbol{a = 1}$, atunci $\boldsymbol{rang\ A = 2}$, alegand $\Delta_p = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$, la intersectia liniilor 2 si 3 si a coloanelor 2 si 3. Mai departe,

$\Delta_c = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & -2 \end{vmatrix} = 0$ (liniile 1 si 2 sunt egale). Astfel, $\boldsymbol{rang\ A = rang\ \overline{A} = 2}$, deci sistemul este compatibil simplu nedeterminat.

Daca $\boldsymbol{a = -3}$, atunci $\boldsymbol{rang\ A = 2}$**,** luand acelasi Δ_p ca la situatia a = 1. In acest caz, $\Delta_c = \begin{vmatrix} 1 & 1 & -1 \\ -3 & 1 & 3 \\ 1 & -1 & -2 \end{vmatrix} = -4 \neq 0$, deci

$\boldsymbol{rang\ \overline{A} = 3 \neq rang\ A}$ **.**

Asadar, $\boldsymbol{a = -3}$ este singura valoare a lui a pentru care sistemul dat este incompatibil.

(iii) Conditia de progresie aritmetica $y = \frac{x+z}{2} \Leftrightarrow \boldsymbol{x} - 2y + z + 0,$ impreuna cu sistemul dat, conduce la sistemul liniar de 4 ecuatii cu 3 necunoscute x, y, z

$$\begin{cases} ax + y + z = -1 \\ x + ay + z = -\mathrm{a} \\ x + y - z = -2 \\ \boldsymbol{x} - 2y + z = 0 \end{cases}$$

unde $\boldsymbol{a}$ este un parametru real.

Fie $\boldsymbol{\mathcal{A}} = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$ matricea sistemului si

$\overline{\boldsymbol{\mathcal{A}}} = \begin{pmatrix} a & 1 & 1 & -1 \\ 1 & a & 1 & -a \\ 1 & 1 & -1 & -2 \\ 1 & -2 & 1 & 0 \end{pmatrix}$ matricea sa extinsa.

Observam ca $\boldsymbol{\mathcal{A}} \in \mathcal{M}_{4,3}(\mathbb{R})$, deci

$$\boldsymbol{rang}(\boldsymbol{\mathcal{A}}) \leq \mathbf{3} \qquad (\mathbf{1}) \,.$$

Pe de alta parte, conditia de compatibilitate a sistemului (impusa de cerintele problemei), anume

$$\boldsymbol{rang}(\boldsymbol{\mathcal{A}}) = \boldsymbol{rang}(\overline{\boldsymbol{\mathcal{A}}}) \qquad (\mathbf{2}) \,,$$

impreuna cu relatia $(\mathbf{1})$, conduce la egalitatea

$$\boldsymbol{det}(\overline{\boldsymbol{\mathcal{A}}}) = \mathbf{0} \qquad (\mathbf{3}) \,.$$

Intr-adevar, daca $\det(\bar{\mathcal{A}}) \neq 0$, atunci $\operatorname{rang}(\bar{\mathcal{A}}) = 4$, ceea ce, alaturi de relatiile **(2)** si **(1)** , furnizeaza contradictia

$\boldsymbol{rang}(\boldsymbol{\mathcal{A}})$ = **4** $\leq$ **3**.

Sa calculam $\det(\bar{\mathcal{A}})$. Scazand linia 1 din liniile 2 si 4, apoi adunand linia 1 la linia 3, obtinem

$$\boldsymbol{det}(\bar{\boldsymbol{\mathcal{A}}}) = \begin{vmatrix} a & 1 & 1 & -1 \\ 1-a & a-1 & 0 & 1-a \\ 1+a & 2 & 0 & -3 \\ 1-a & -3 & 0 & 1 \end{vmatrix} =$$

$$(-1)^{1+3}.1.\begin{vmatrix} 1-a & a-1 & 1-a \\ 1+a & 2 & -3 \\ 1-a & -3 & 1 \end{vmatrix} = (1-a)\begin{vmatrix} 1 & -1 & 1 \\ 1+a & 2 & -3 \\ 1-a & -3 & 1 \end{vmatrix}$$

Adunand coloana 3 la coloana 2 si scazand-o din coloana 1, primim:

$$\boldsymbol{det}(\bar{\boldsymbol{\mathcal{A}}}) = (1-a)\begin{vmatrix} 0 & 0 & 1 \\ 4+a & -1 & -3 \\ -a & -2 & 1 \end{vmatrix} = (a-1)(8+3a).$$

Astfel , ecuatia **(3)** are doua solutii, $a = 1$ si $a = -\frac{8}{3}$.

Se observa ca ambele solutii convin, deoarece in ambele situatii este indeplinita relatia (2).

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